D-branes 101

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ITP Miniprogram: Geometry & Duality

January 1998

www.itp.ucsb.edu

1. Introduction and overview

Dbranes combine methods of string theory and gauge theory in an elegant way.

An enormous amount of work has been done on Dbranes and string duality. ($\sim$ 1000 – 2000 papers on hep-th) Our goal in these lectures will be very modest:

The goal of these lectures is to show how the theory of D-branes makes the ADHM construction of instantons on $\mathbb{R}^4$ extremely natural and physical. In particular they give new and interesting answers to the question: "What happens when an instanton becomes small?"

1.1. Warning

Warning to physicists: These lectures are primarily aimed at mathematicians who have not thought much about Dbranes before. I have tried to stress points which most physicists would take for granted, perhaps at the risk of being ridiculously pedantic.

Also, many important topics are omitted. I have completely ignored Type I = heterotic, orientation projections, orientifolds, etc.

1.2. Sources

We often will be following closely the development of

1. Chaudhuri, Johnson, Polchinski, "Notes on D-branes,"
2. J. Polchinski, "TASI lectures on D-branes,"

Other recent relevant reviews:

1. Townsend
2. Youm, Dbranes and black holes
3. Banks, Matrix theory

All available on hep-th.

Because I am determined *not* to start writing a review article I have quite consciously omitted almost all references and names. The results are due to the usual suspects. The references can be found in the above reviews.

$\mathcal{A}(M;E)$ the space of connections on a vector bundle $E \to M$.

$\mathcal{A}_{(\mathcal{B},\mathcal{B}')}$ A conformal field theory associated with boundary conditions.

$\alpha, \beta, \dots$ Generic notation for spinor indices. Often for **16** of $Spin(1,9)$.

$\mathcal{B}_p$: A coordinate coordinate plane in $\mathbb{R}^{1,s}$ with p spatial directions. Usually, the spatial locus of a p-brane.

d_T: Number of transverse dimensions to a brane. For a p-brane in D dimensions $d_T = D - p - 1$.

D: Dimension of spacetime, or Dirichlet boundary condition.

$\epsilon, \tilde{\epsilon}$: MW spinors of $Spin(1,9)$, or of various subgroups.

$\mathcal{F}_p$: A Fock space based on a highest weight state $|p\rangle$

$\eta(\mathcal{W} \hookrightarrow \mathcal{S})$ a δ-function supported representative of the Poincaré dual of $\mathcal{W}$ in $\mathcal{S}$.

λ: A generic symbol for an element of $Hom(V,W)$, with V, W finite dimensional vector spaces, usually Chan-Paton spaces.

ℓ_s The string scale.

M: A Lorentz index, running $M = 0, 1, \dots, s$

$\mathbf{1_N}$ The $N \times N$ unit matrix.

$\mathbb{N}$: a number operator in a CFT with spectrum the natural numbers including 0.

$\Omega^k(M)$: The space of k-forms on a manifold M.

p: Spatial dimension of a brane. The worldvolume has dimension $p+1$. Also, a momentum vector as in p^μ.

$\mathcal{P}(1,s)$ The Poincaré group of $\mathbb{R}^{1,s}$

Ψ: Generic symbol for a quantum state, in a string field theory or in a QFT.

$\mathbb{R}^{1,s}$ Minkowski space of signature $\eta_{MN} \equiv Diag\{-1, +1^s\}$

s: Dimension of space.

$\mathcal{S}$: Spacetime.

$\mathcal{SP}(1,s)$: The minimal superpoincaré algebra of $\mathbb{R}^{1,s}$.

Σ: A worldsheet

$u(N)$: The Lie algebra of $N \times N$ antihermitian matrices, the adjoint representation of $U(N)$, the $N \times N$ unitary group.

$\mathcal{W}_{p+1}$: The worldvolume of a p-brane: $\mathcal{W}_{p+1} = \mathcal{B}_p \times \mathbb{R}$.

$YM(\mathcal{B}_p)$: A low energy effective Yang-Mills theory on a D-brane located at $\mathcal{B}_p$. A superior notation is $YM(\mathcal{W}_{p+1})$, and, when we wish to emphasize the Chan-Paton bundle, $YM(\mathcal{W}_{p+1}, E)$.

2. The open bosonic string, and its spacetime interpretation

The physics of Dbranes is under the best control in supersymmetric theories. However, several important ideas can be explained without the extra complications of fermions. So, to begin we will set all fermions, on worldsheet and spacetime to zero.

2.1. Free open string in Minkowski spacetime

Interval $I = [0, \pi]$
Open string: $X : I \to \mathbb{R}^{1,s}$
Strip Σ: $0 \le \sigma^1 \le \pi \qquad -\infty \le \sigma^2 < \infty$
Minkowski signature ws: $\sigma^2 = i\tau$.

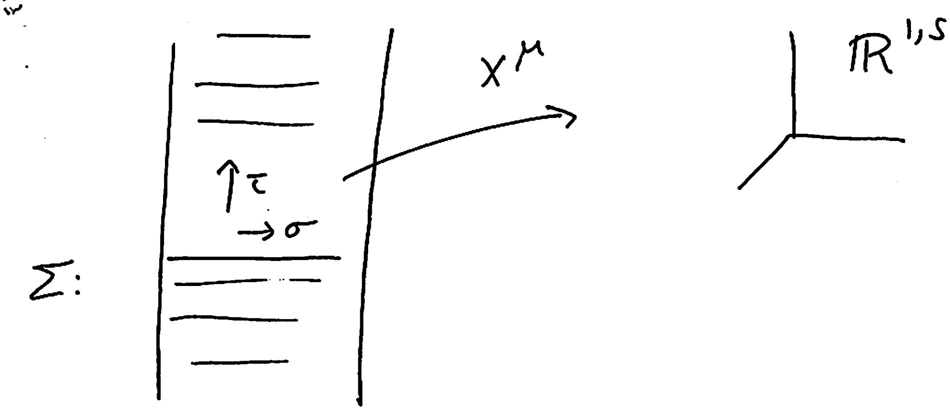


Fig. 1: Map of the strip to spacetime

Action:

$$S = \frac{1}{4\pi\ell_s^2}\int_\Sigma d^2\sigma\, \partial_a X^\mu \partial_a X_\mu = \frac{1}{4\pi\ell_s^2}\int_\Sigma \langle \partial X, \bar\partial X\rangle$$

The coordinates X^μ have length, hence we must introduce a fundamental scale, ℓ_s. The corresponding energy scale is M_s.

The variational problem is not defined until we choose boundary conditions:

$$\delta S = -\frac{1}{2\pi\ell_s^2}\int_\Sigma d^2\sigma\, \delta X^\mu \partial^2 X_\mu + \frac{1}{2\pi\ell_s^2}\int_{\partial\Sigma} d\sigma\, \delta X^\mu \partial_1 X_\mu$$

⇒Two solutions on each component of the boundary:

Neumann: $\partial_\sigma X^\mu| = 0$

Dirichlet: $X^\mu| = 0$

For each coordinate direction we have 2×2 choices.

For the moment make the choice (N, N).

2.2. *Conformal field theory*

2.2.1. Quantization

Σ is conformal to upper half-plane: $z = e^{i\sigma^1 + \sigma^2}$

Propagation in UHP → radial propagation

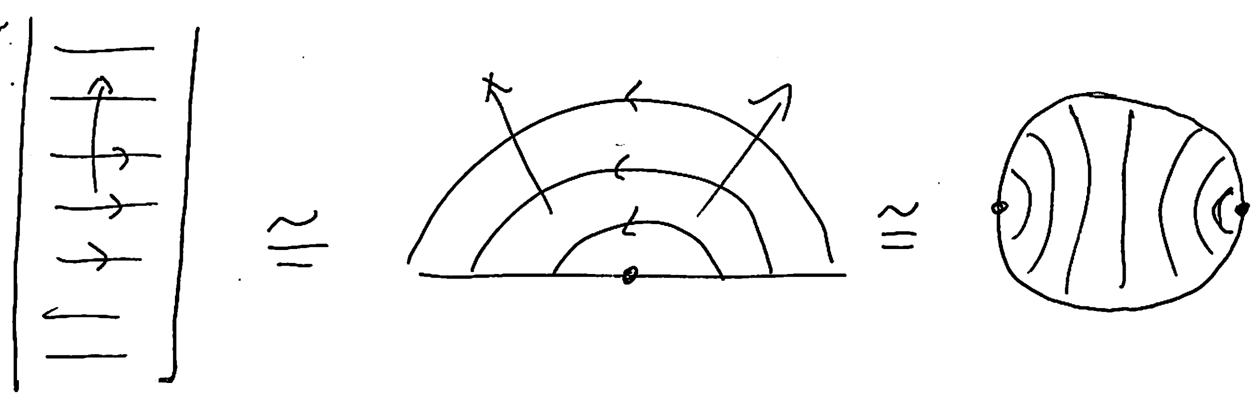


Fig. 2: Radial propagation in the upper half plane.

(N, N) Oscillator expansion:

$$X = x + ip\tau + i\sum_{m\neq 0}\frac{\alpha_m}{m}e^{im\tau}\cos m\sigma^1$$
$$= x + p(\log z + \log \bar{z}) + i\sum_{m\neq 0}\frac{\alpha_m}{m}(z^{-m} + \bar{z}^{-m})$$

Quantization requires representation of the Heisenberg algebra:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}$$
$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \tag{2.1}$$

Choose heighest weight representation ⇒Fock space:

$$\alpha_n^\mu|p\rangle = 0 \qquad n > 0$$
$$\alpha_0^\mu|p\rangle = p^\mu|p\rangle \qquad n > 0$$
$$\mathcal{F}_p \equiv Span\{\prod \alpha_{-n_i}^{\mu_i}|p\rangle\}$$
$$\mathcal{A} = \oint d^D p\, \mathcal{F}_p$$

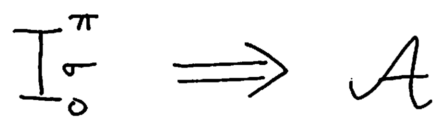
2.2.2.Vertex operators, modular functor

One can formulate CFT for open strings axiomatically in the way proposed by Segal.

Objects: Intervals

Morphisms: Surfaces:

Functor: Intervals $\rightarrow \mathcal{A}$



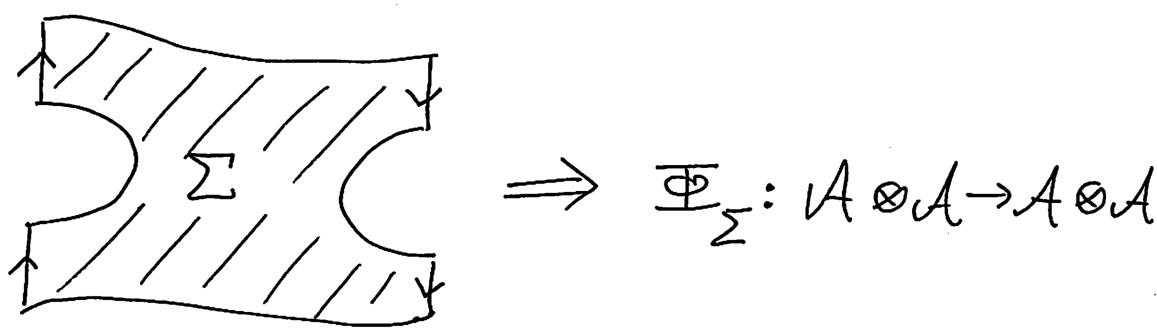


Fig. 3: Surfaces as morphisms

By conformal invariance the same information is carried by Vertex operator correlation functions on the disk or the upper halfplane:

2.2.3.Digression: CFT with a boundary

CFT representation of a disk amplitude with

$W_i(x)$, open string VO's inserted on the boundary.

$V_i(z) \otimes \bar{V}_i(\bar{z})$ VO's inserted in the bulk.

transform the disk amplitude into sphere amplitude of a *chiral CFT*:

Doubling trick: equivalent to single oscillator on complex plane.

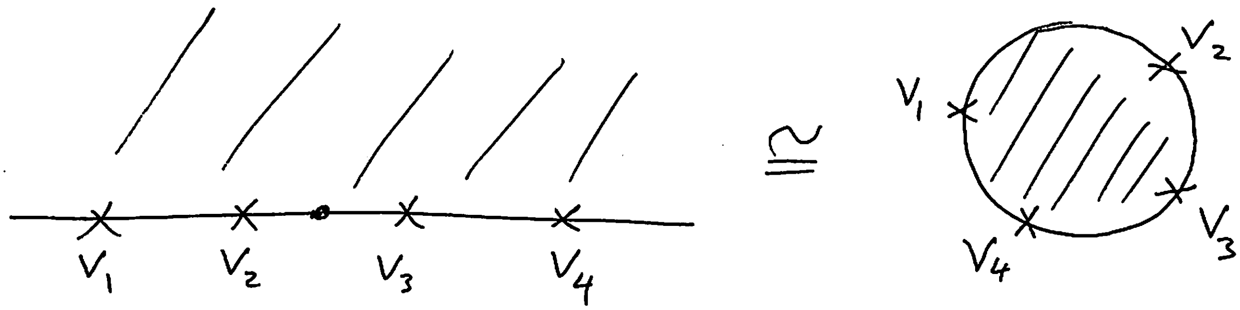


Fig. 4: Vertex operators on the disk.

Vertex operators inserted on edge → chiral VO's on the line

Example:

$$\langle X(z)X(w)\rangle = -\log \frac{|z-w|^2}{|z-\bar{w}|^2} \qquad D$$

$$\langle X(z)X(w)\rangle = -\log |z-w|^2 - \log |z-\bar{w}|^2 \qquad N$$

Remark: One can consider general classes of conformally invariant boundary conditions. Boundary CFT: [1] A recent paper applying it to branes: [2]

3. Spacetime Interpretation

The global symmetry of the theory is $\mathcal{P}(1,s)$, the Poincaré group.

The negative signature of the timelike oscillators ⇒problems with unitarity.

⇒BRST cohomology

$$\mathcal{H} = H_d^*\left(\oint dp\mathcal{F}_p \otimes \Lambda_{bc}\right)$$

d: Chevalley-Eilenberg Lie algebra cohomology differential for the Virasoro algebra.

Claim: This is a completely reducible unitary representation of the Poincaré group $\mathcal{P}(1,s)$.

Proof: Use the grading by level $\mathbb{N}$. At each level we get an induced representation induced from a finite dimensional representation of the little group.

- *In field theory: Unitary irreps ↔ particles in the spacetime* $\mathbb{R}^{1,s}$

Spectrum:

$$0 = L_0 - 1 = \ell_s^2 p^2 - 1 + \mathbb{N} \qquad N \geq 0$$

$$\rightarrow$$

$$E^2 - \vec{p}^2 = m^2 = (\mathbb{N} - 1)\ell_s^{-2}$$

3.1. Massless modes of the gauge theory

Of greatest importance are the spactime massless modes, associated with the CFT states at level $\mathbb{N} = 1$:

$$\epsilon_\mu \alpha_{-1}^\mu |p\rangle \in \mathcal{H} \qquad \leftrightarrow \qquad \epsilon_\mu \partial X^\mu e^{ip\cdot X}$$

Kerd: $\Rightarrow p \cdot \epsilon = 0$

Imd: $\Rightarrow \epsilon \sim \epsilon + xp$.

$\Rightarrow$A massless vector boson representation of $\mathcal{P}(1, s)$.

- *In local field theory particles are described by local quantum fields*

The field describing the massless mode of the open string is the Maxwell field $A_\mu(x)$ with gauge invariance $A \rightarrow A + d\lambda$.

3.2. The S-matrix

An important physical quantity is the S-matrix:

$$S : \oplus_N S^N \mathcal{H} \rightarrow \oplus_N S^N \mathcal{H}$$

It is obtained by using The vertex operator correlation functions to define a certain measure on the moduli space of surfaces and integrating. One also has to sum over orderings.

As long as we work at tree level this can be rigorously defined by analytic continuation in momentum.

3.2.1.The spacetime action

Of course, there is an infinite tower of particles. One might try to describe them by an infinite set of quantum fields. For this infinite set of fields we might try to formulate an action principle which reproduces the above S-matrix.

Describing the spacetime field theory interactions of these particles is a difficult problem, still unsolved today.

For the tree level open string there is an elegant proposal of Witten's.

Definition. The off-shell statespace is:

$$\mathcal{A} \equiv \oint dp \mathcal{F}_p \otimes \Lambda_{bc}$$

A string field is:

$$\Psi \in \mathcal{A}$$

The ingredients we need:

a.) A graded differential: $d : \mathcal{A} \to \mathcal{A}$, $d^2 = 0$, $\deg(d) = 1$.

b.) A (noncommutative) product: $* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

c.) An integration $\int : \mathcal{A} \to \mathbb{C}$.

Action:

$$\int \Psi * d\Psi + \frac{2}{3} \Psi * \Psi * \Psi$$

3.3. The Chan-Paton construction

The three ingredients (a, b, c) allow the modification:

$$\mathcal{A} \to \mathcal{A} \otimes Mat_N(\mathbb{C})$$

where $Mat_N(\mathbb{C})$ is the algebra of $N \times N$ complex matrices.

The modification of the correlation functions is:

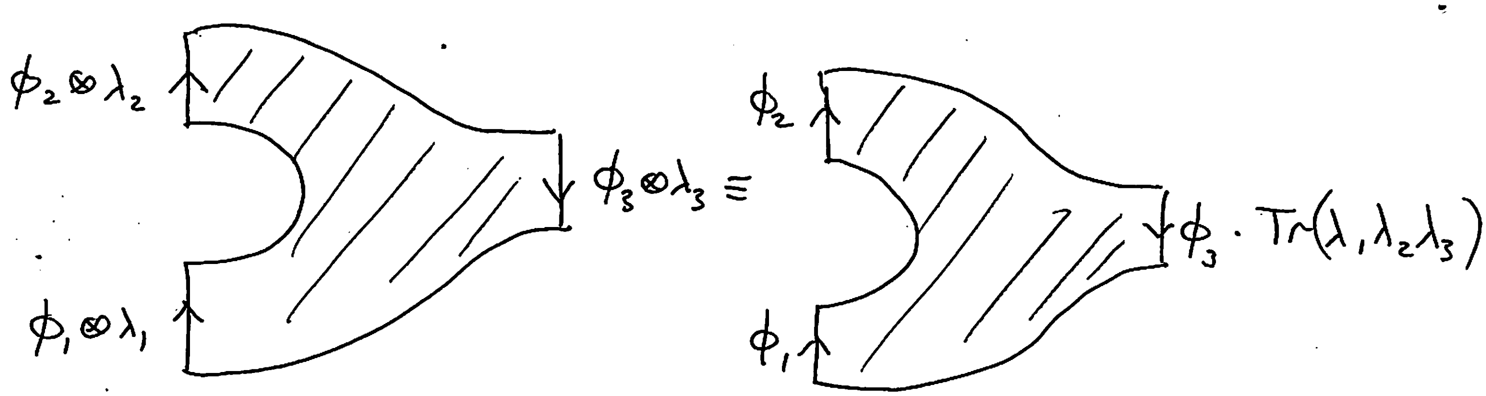


Fig. 5: Product of vertex operators on a disk times a trace.

Remark. $\mathcal{A}$ represents *oriented* open strings and hence has an anti-hermitian involution. In string theory we impose a reality condition. In terms of the CP construction this requires the replacement:

$$Mat_N(\mathbb{C}) \to u(N)$$

3.3.1.CP ⇒Nonabelian gauge theory

The Chan-Paton construction modifies the string field theory action by:

$$\int Tr\left[\Psi * d\Psi + \frac{2}{3}\Psi * \Psi * \Psi\right]$$

When we keep only the massless degrees of freedom - i.e. take the low energy effective theory - we get the nonabelian gauge theory action:

$$S = \frac{1}{g_{YM}^2}\int\left[TrF \wedge *F + \mathcal{O}(p^6\ell_s^2)\right]$$
$$F = dA + \frac{1}{2}[A, A]$$

Remarks:

1. Leading term follows from folk theorem: Locality + Lorentz+ Gauge ⇒YM action, at leading order.
2. The three ingredients are provided any *Frobenius algebra*

4. D & N boundary conditions, and their first spacetime interpretation

4.1. Dirichlet boundary conditions

Now let us return to the choice of D or N at each boundary of $I = [0, \pi]$.
For each coordinate direction μ, choose D,N at $\sigma = 0, \pi \Rightarrow$4 choices.

4.1.1.Pictorial interpretation

Let us interpret these boundary conditions pictorially:

	0,1,. . . , p	p+1,. . . , s
$\sigma = 0$	N	D
$\sigma = \pi$	N	D

Table 1: Some boundary conditions

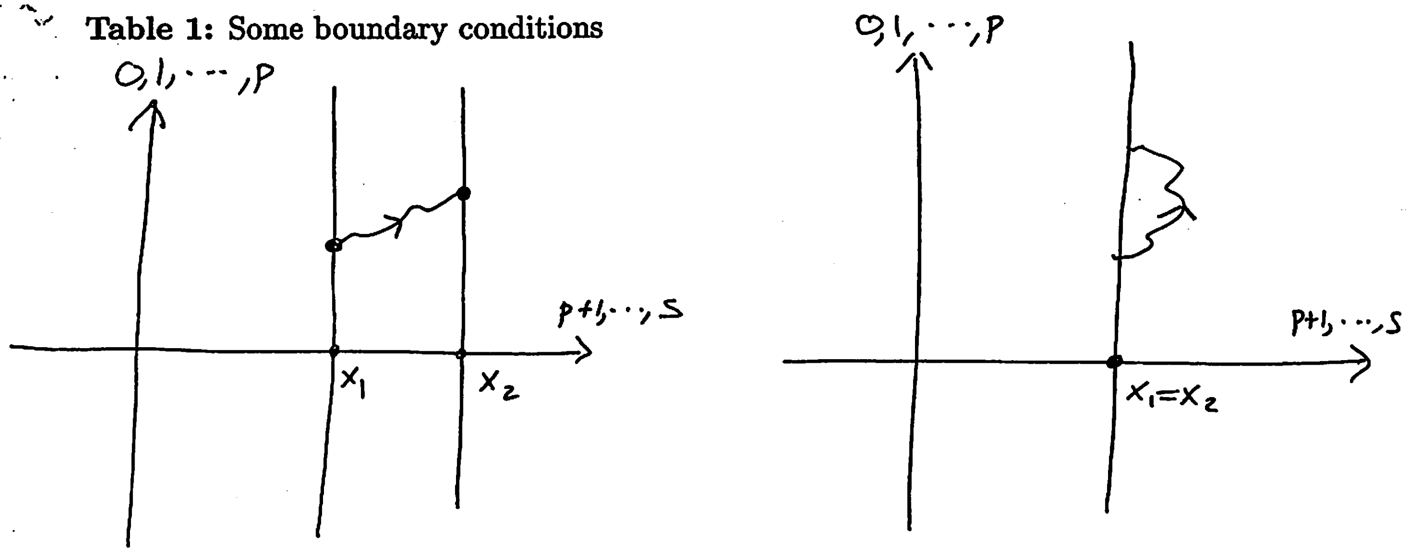


Fig. 6: $p+1$ coordinates have boundary conditions of type (N, N). The remaining have boundary conditions of type (D, D). RHS picture is an important special case.

	0,1,. . . , p	p+1,. . . , p'	p'+1,. . . , s
$\sigma = 0$	N	D	D
$\sigma = \pi$	N	N	D

Table 2: Mixed boundary conditions

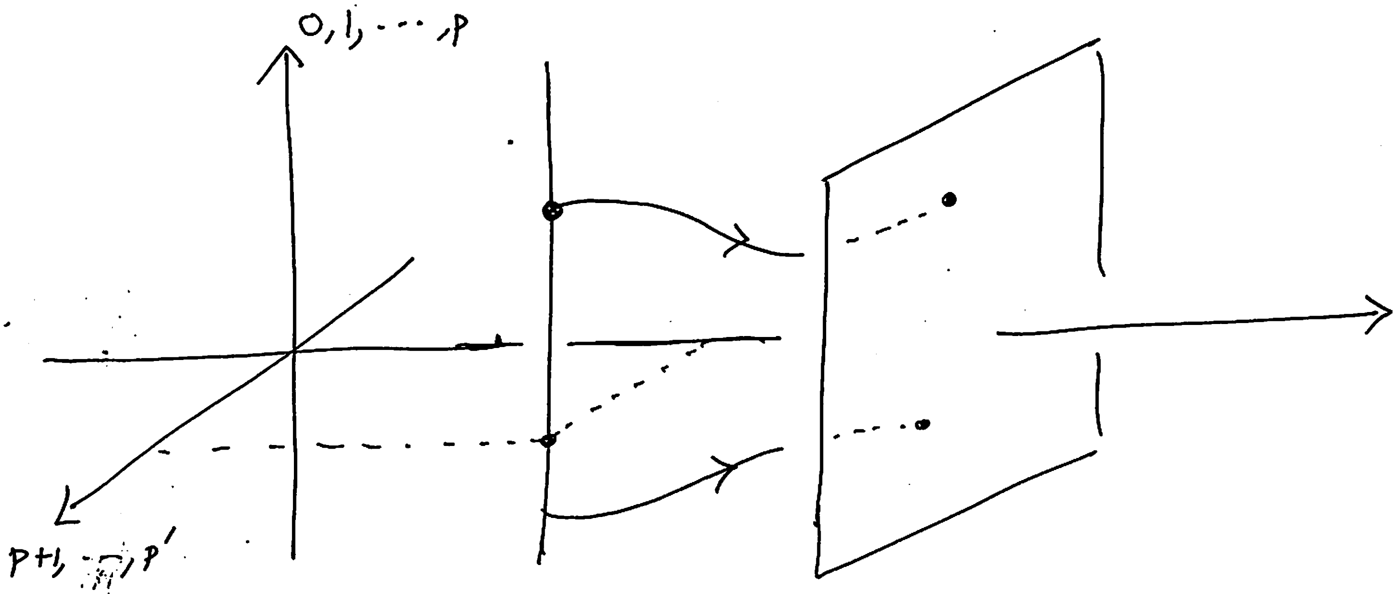


Fig. 7: Typical strings in a sector with mixed boundary conditions.

Conclusion: *The choices of boundary conditions are equivalent to a choice of an ordered pair $(\mathcal{B}_1, \mathcal{B}_2)$ of coordinate coordinate planes in space. We denote the corresponding CFT by $\mathcal{A}_{(\mathcal{B}_1,\mathcal{B}_2)}$.*

4.1.2. Oscillators

Now, let us try to understand these CFT's $\mathcal{A}_{(\mathcal{B}_1,\mathcal{B}_2)}$.

(N, N): These coordinates have a momentum zeromode:

$$X = x + ip\tau + i \sum_{m\in\mathbb{Z}-0} \frac{\alpha_m}{m} e^{im\tau} \cos m\sigma^1$$
$$= x + p(\log z + \log \bar{z}) + i \sum \frac{\alpha_m}{m} (z^{-m} + \bar{z}^{-m})$$

(D, D): These coordinates have a position zeromode:

$$X = x_1 + (x_2 - x_1)\frac{\sigma^1}{\pi} + i \sum_{m\in\mathbb{Z}-0} \frac{\alpha_m}{m} e^{im\tau} \sin m\sigma^1$$
$$= x_1 + (\Delta x)(\log z - \log \bar{z}) + i \sum_{m\in\mathbb{Z}-0} \frac{\alpha_m}{m} (z^{-m} + \bar{z}^{-m})$$

(D, N): neither momentum nor position zm.

$$X = x + i \sum_{m\in\mathbb{Z}} \frac{\alpha_{m+\frac{1}{2}}}{m+\frac{1}{2}} e^{i(m+\frac{1}{2})\tau} \sin(m+\frac{1}{2})\sigma^1$$
$$= x + i \sum \frac{\alpha_{m+\frac{1}{2}}}{m+\frac{1}{2}} (z^{-m+1/2} + \bar{z}^{-m+1/2})$$

(N,D) : switch: $\sin(m+\frac{1}{2})\sigma^1 \to \cos(m+\frac{1}{2})\sigma^1$

4.2. Dynamics on the branes

To give a spacetime interpretation we consider now the BRST cohomology:

$$\mathcal{H}_{(\mathcal{B}_1,\mathcal{B}_2)} \equiv H_d^*\Big(\mathcal{A}_{(\mathcal{B}_1,\mathcal{B}_2)} \otimes \Lambda\Big)$$

in a way analogous to what we did when all coordinates had (N, N) boundary conditions.

For this section we take $\mathcal{B}_1 = \mathcal{B}_2$. We come back to $\mathcal{B}_1 \neq \mathcal{B}_2$ later.

4.2.1. Oscillator quantization

Take a single coordinate plane at

$$\mathcal{B} := \{X : X^a = X_0^a \qquad a = p+1, \ldots s\}$$

That is, the bc's are:

$$\begin{array}{lll} (N,N) & X^\mu, & \mu = 0,1,\ldots p \\ (D,D) & X^a & a = p+1,\ldots,s \end{array}$$

Referring to the above oscillator expansions the Heisenberg algebra has hardly changed. The only new point is that we now must take a highest weight state $|k^\mu, X_0^a\rangle$ defined by:

$$\begin{aligned} &\alpha_n^{\mu,a}|k^\mu, X_0^a\rangle = 0 \qquad n > 0 \\ &\alpha_0^\mu|k^\mu, X^a\rangle = k^\mu|k^\mu, X^a\rangle \qquad \mu = 0,1,\ldots,p \end{aligned}$$

This hwstate generates the Fock space $\mathcal{F}_{k,X_0}$, so we have arrived at our conclusion: *The CFT is*

$$\mathcal{A}_{(\mathcal{B},\mathcal{B})} = \oint d^{p+1}k \ \mathcal{F}_{k,X_0}$$

Note that the CFT statespace has a "modulus" - we can speak of a fixed position, and we are automatically considering families of statespaces over spacetime. [1]

[1] Physically, the branes we are going to describe are big and heavy, and can be localized.

4.3. *Spacetime fields on the Dbrane*

The Dp-brane clearly breaks the spacetime symmetry

$$\mathcal{P}(1,s) \supset \mathcal{P}(1,p) \times SO(d_T)$$

The on-shell statespace

$$\mathcal{H}_{(\mathcal{B},\mathcal{B})} = H_d^*\left(\int d^{p+1}k \quad \mathcal{F}_{k,X_0} \otimes \Lambda_{bc}\right)$$

will form a completely reducible unitary representation of $\mathcal{P}(1,p) \times SO(d_T)$.

Interpretation:

The modes of these strings are particles confined to the p-dimensional coordinate plane $\mathcal{B}$. $SO(d_T)$ is a global symmetry of the theory.

4.3.1.Massless modes

The statespace at level $N = 1$ is spanned by:

$$\alpha_{-1}^{\mu}|k^{\mu}, X_0^a\rangle \qquad \mu = 0, \ldots, p$$
$$\alpha_{-1}^{a}|k^{\mu}, X_0^a\rangle \qquad a = p+1, \ldots, s$$

When we study the gauge dependence

$$\epsilon^{\mu} \to \epsilon^{\mu} + xk^{\mu}$$
$$\epsilon^{a} \to \epsilon^{a}$$

Thus, we have a connection tangent to the brane and a gauge invariant scalar particle in the fundamental of $SO(d_T)$. That is, the global symmetry group $\mathcal{P}(1,p) \times SO(d_T) \times U(N)$ has particle representation:

$$(Ind(p-1; 1; u(N)) \oplus (Ind(1); d_T; u(N))$$

The worldsheet-spacetime correspondence is:

W	**S**	
$\alpha_{-1}^{\mu}\|k^{\mu}, X_0^a\rangle \otimes \lambda$	$(A_{\mu}(x))^i{}_j$	$A \in \mathcal{A}(\mathcal{W}_{p+1}; u(N))$
$\alpha_{-1}^{a}\|k^{\mu}, X_0^a\rangle \otimes \lambda$	$(\phi_a(x))^i{}_j$	$\phi_a \in \Omega^0(\mathcal{W}_{p+1}; u(N))$

$\lambda \in u(N)$, $i, j = 1, \ldots, N$ are CP indices, $x \in \mathcal{W}_{p+1} = \mathcal{B}_p \times \mathbb{R}$.

Result: the massless field content is the Yang-Mills-Higgs system obtained by dimensional **reduction** *of the D-dimensional Yang-Mills theory along the directions* X^a*. We will denote this* $p+1$*-dimensional QFT* $YM(\mathcal{B}_p)$ *or sometimes* $YM(\mathcal{W}_{p+1})$*. It is governed by the low energy effective action:*

$$S \sim \frac{1}{g_{YM}^2} \int TrF \wedge *F + Tr(D\phi)^2 + \sum_{a<b} \mathrm{Tr}([\phi_a, \phi_b])^2 + \cdots$$

Idea of Proof: The vertex operator calculations leading to Nonabelian gauge theory carry over almost unchanged because the oscillator spectrum is almost unchanged. Only the momentum zeromodes have been changed.

5. The brane and the bulk

The spacetime interpretation from the previous section is not really satisfactory.

Now, if we use the orthogonal decomposition of the Lie algebra:

$$u(N) = su(N) \oplus u(1)$$

we see that the $u(1)$ degree of freedom decouples to give a free maxwell field + free scalar.

$$\int F \wedge *F + \sum_a d\phi_a \wedge *d\phi_a$$

In this free scalar theory the vacuum state is specified by the zeromode $(\phi_a)_0$....

5.0.1.The meaning of the zeromode: $(\phi_a)_0$

The $\phi_a(x)$ don't transform in the $U(1)$ theory. The zeromode is gauge invariant and has a physical meaning: It is the *position of the brane in the a-direction.* $(\phi_a)_0$ is conjugate to the momentum of the brane in the a^{th} direction. Put another way, the massless bosons $\phi^a(x)$ on $\mathcal{B}$ are the Goldstone bosons for the breaking of the *global* symmetry of the CFT corresponding to translations in the a^{th} direction.

Another proof is to consider the Polyakov path integral representation of the morphism $\Phi_\Sigma : \mathcal{A} \to \mathcal{A}$ in the case where $\Sigma = [0, \pi] \times [0, T]$. In the presence of nontrivial fields (A, ϕ) on $\mathcal{W}_{p+1}$ the path integral has the form:

$$\int dX(\sigma, \tau) e^{-\int \langle \partial X, \bar{\partial} X \rangle} Tr_{\mathbb{C}^N} \left(Pexpi \int_{\partial\Sigma} X^*(A) + \partial_n X^a \phi_a \right)$$

now we note that this path integral has a symmetry:

$$X_0^a \to X_0^a + \delta X_0^a$$
$$\phi_a \to \phi_a - \delta X_0^a 1$$

5.0.2.D branes wiggle

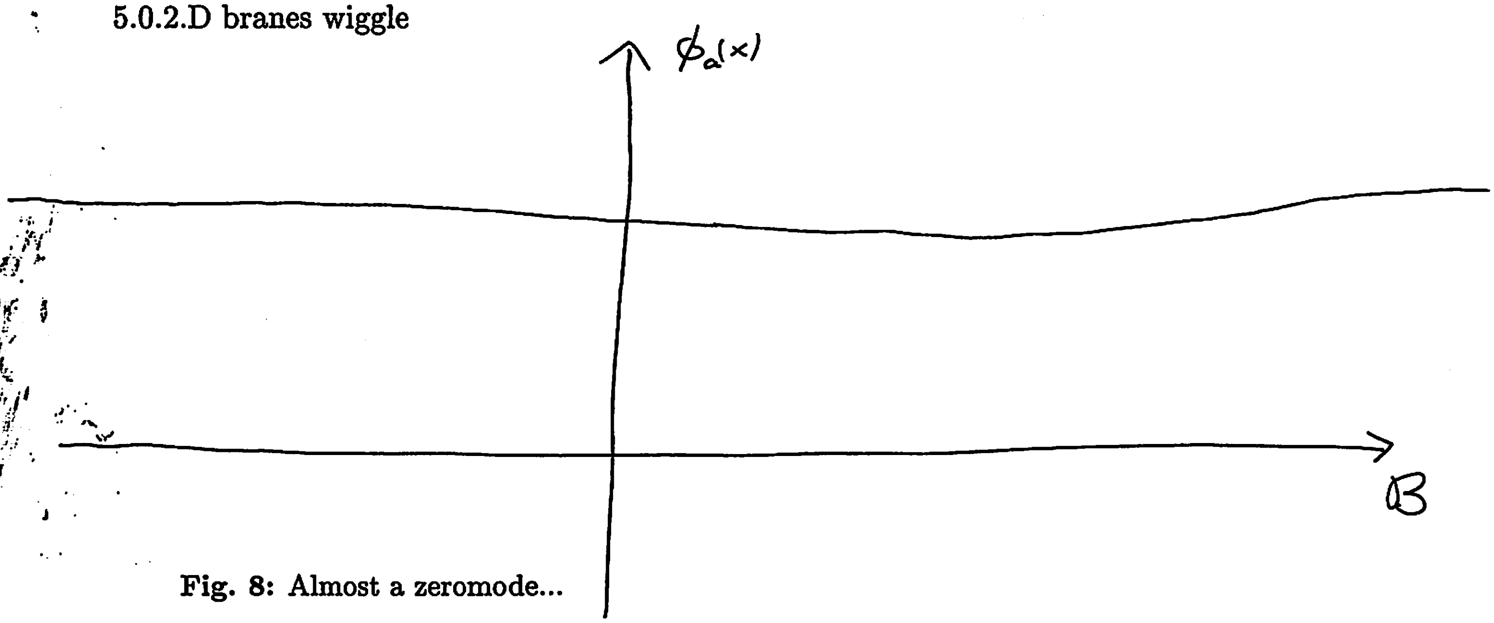


Fig. 8: Almost a zeromode...

Therefore: Local variations of the position are oscillations of the brane - it wiggles.

Thus the coordinate plane $\mathcal{B}$ will turn out to be the location of a *dynamical object* which can move in the ambient space. These dynamical objects are called "Dp-branes."

Example A good example to keep in mind is the way you would describe the effective theory of the ocean: The height of the waves is a scalar field in a 2+1 dimensional field theory defined on (roughly) $\mathcal{B} = S^2$.

5.1. A theory of the brane itself?

One could try to formulate a phsyical theory in $p+1$ dimensions based solely on the string sector $\mathcal{A}_{(\mathcal{B},\mathcal{B})}$.

Indeed, people have speculated if we ourselves might not be "confined" to a 3-brane embedded in a larger space...

At least in the string model described here that runs into a major difficulty - the $\mathcal{S}$-matrix based on $\mathcal{A}_{(\mathcal{B},\mathcal{B})}$ alone is not unitary.

Related to this: energy and momentum will not be conserved:

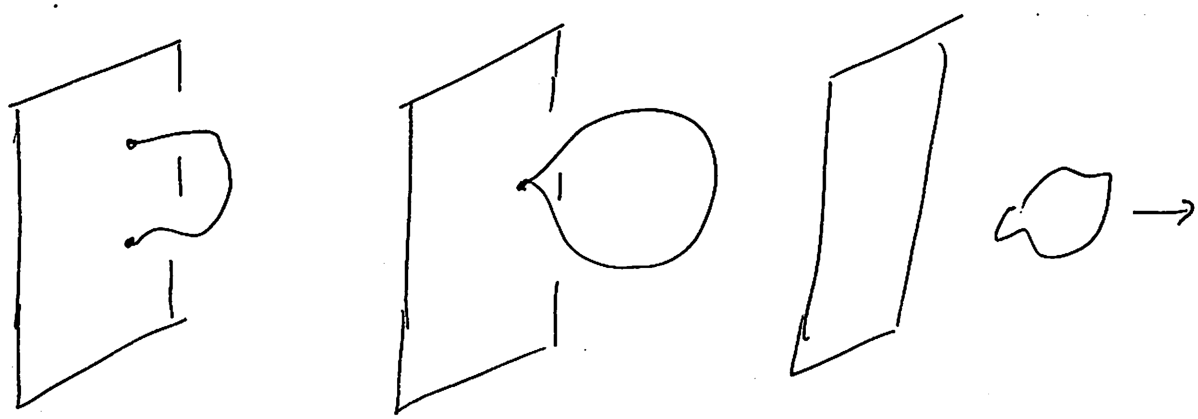


Fig. 9: A string escapes from the Dp-brane into space.

The quantum field theorist doing scattering experiments on $\mathcal{B}$ thus sees an apparent lack of unitarity:

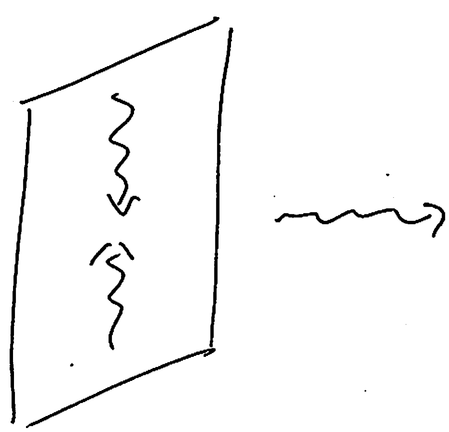


Fig. 10: Loss of unitarity in the brane QFT.

Mathematically, the "good functor" should include all Riemann surfaces:

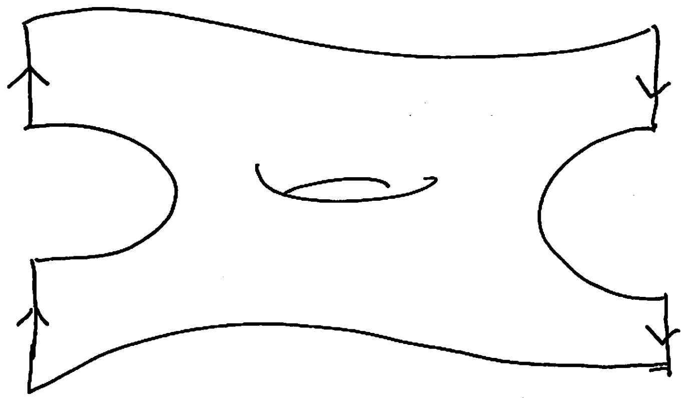


Fig. 11: Other important morphisms.

5.2. The closed string sector

We are simply discovering the closed string sector based on the oscillator expansion:

$$X^\mu = x^\mu + \alpha_0^\mu \tau + i \sum_m \frac{1}{m}[\alpha^\mu z^{-m} + \tilde{\alpha}^\mu \bar{z}^{-m}]$$

The corresponding statespace is

$$\mathcal{A}_{closed} = \int d^D p \mathcal{F}_p \otimes \tilde{\mathcal{F}}_p \otimes \Lambda_{b,c,\bar{b},\bar{c}}$$

Of greatest interest are the massless states:

$$e^{ipX} \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu \qquad \epsilon \in (T^* \mathcal{S})^{\otimes 2}$$

These are conveniently separated into irreducible representations, together with the corresponding spacetime fields:

W	**S**	
$\epsilon \in S^2(T^*\mathcal{S})_0$	$g_{MN}(x)$	$g \in MET(\mathcal{S})$
$\epsilon \in \Lambda^2(T^*\mathcal{S})$	$B_{MN}(x)$	$B \in \Omega^2(\mathcal{S})$
$Tr(\epsilon)$	$\phi(x)$	$\phi \in \Omega^0(\mathcal{S})$

with gauge invariances:

$$\delta_\xi g = \mathcal{L}_\xi g + \mathcal{O}(\ell_s^2) \qquad \xi \in Vect(\mathcal{S})$$
$$\delta_\Lambda B = d\Lambda + \mathcal{O}(\ell_s^2) \qquad \Lambda \in \Omega^1(\mathcal{S})$$

ϕ is gauge invariant and has the interpretation of the string coupling:

$$g_s^2 = \langle e^{2\phi} \rangle$$

The low energy effective action governing these is

$$S = \frac{1}{\ell_s^{D-2}} \int d^D x e^{-2\phi} \Big[\sqrt{g} R + d\phi \wedge * d\phi + dB \wedge * dB + \zeta(3) \ell_s^6 R^4 + \cdots + e^{2\phi} B \wedge Tr(R \wedge R \wedge R \wedge R) + \cdots \Big]$$

A crucial point:

There are *two types of corrections.* Physical processes have a typical energy scale E. We expand physical quantities in the dimensionless parameters g_s and in $E\ell_s$. In general,

a.) QFT breaks down when $E\ell_s \geq 1$. We must replace the Einstein-Hilbert $+\cdots$ action by string theory.

b.) Classicial approximations break down when $g_s \geq 1$.

When these approximations break down answers become singular, but, as Witten said: [2] "In string theory there are no singularities, just surprises." This is certainly the case for the zero-size instanton, as we will see later.

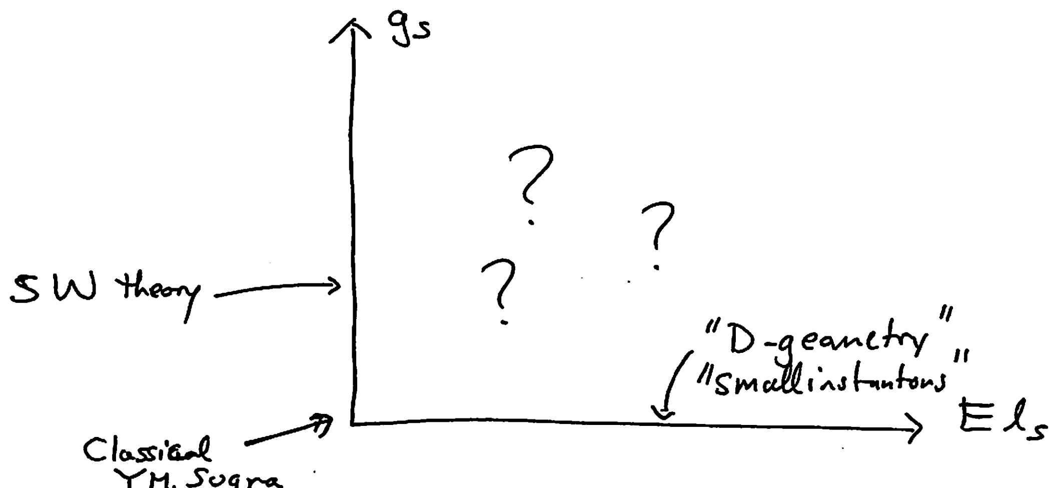


Fig. 12: Ignorance space. E is a typical energy or momentum scale in the low energy derivative expansion.

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[2] at a lecture in Princeton, April 1996, recommending small instantons to mathematicians

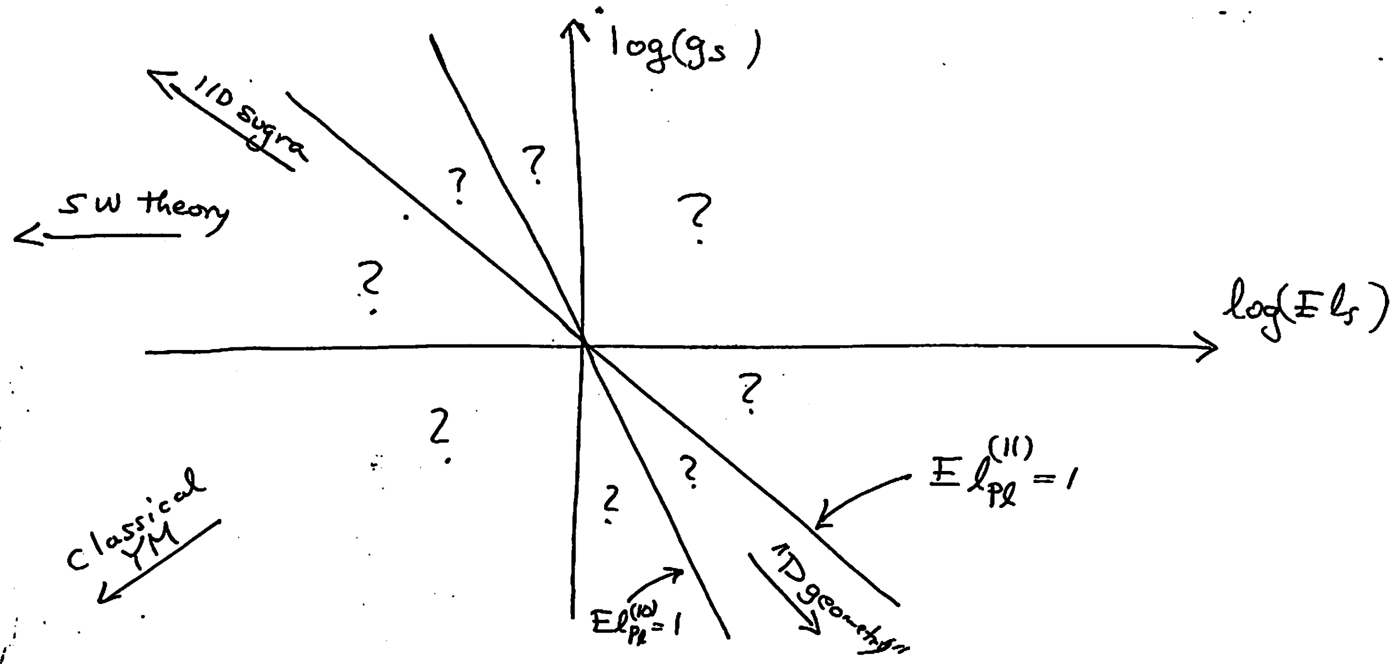


Fig. 13: Actually, PDE's are nontrivial, so the log-log plot is fairer. It also allows a clearer illustration of the sub-stringy regime of "D-geometry"

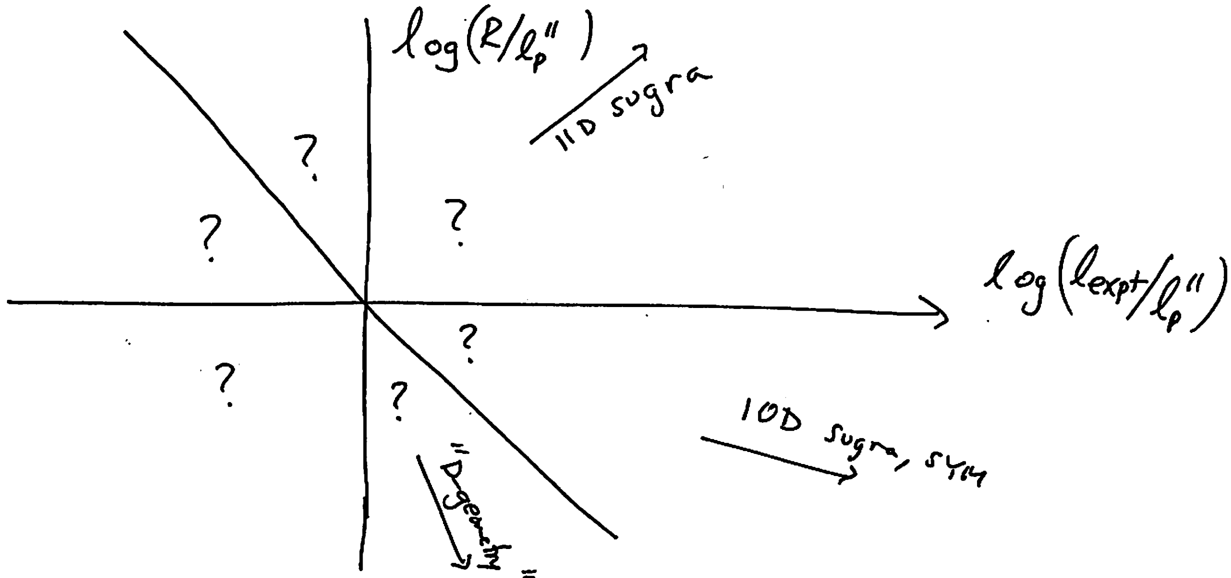


Fig. 14: Ignorance space from the M-theory viewpoint. There is a sub-stringy region which, it is claimed, can be probed by D-branes, describing some kind of "D-geometry."

5.3. The influence of the D-brane on the bulk

The Dp-brane has energy - it creates a gravitational field outside the brane. We are going to describe that field.

5.3.1. How to weigh a D-brane

Now, let us ask:

"What is the mass of a Dp-brane? "

The Dp-brane has energy/spatial volume $= T_p \sim [MASS]^{p+1}$

When you weigh something of mass M_2 you measure the gravitational interaction energy with the earth:

$$\Delta E = -\frac{GM_1M_2}{r}$$

where M_1= mass of the earth.

This is how Polchinski weighed a D-brane.

He calculated the interaction energy bewteen two D-branes using string techniques:

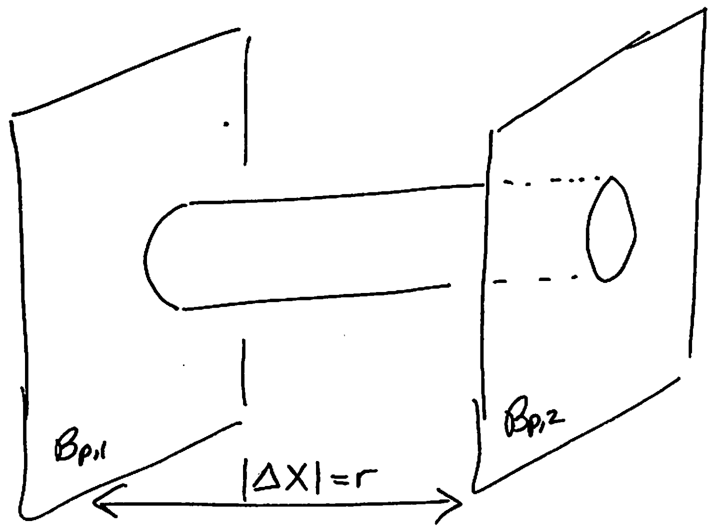


Fig. 15: Parallel D-branes. The leading contribution in the interaction energy is computed from this diagram. The exact energy involves automorphic forms.

The string answer for the interaction energy is:

$$\Delta E \sim \int_0^\infty \frac{dt}{t} t^{-(p+1)/2} e^{-(\Delta X)^2 t/\ell_s^2} f(it)$$

where $f(\tau)$ is an automorphic form (for $\Gamma(2)$).

There are two regimes of interest:

1.) $|\Delta X| \gg \ell_s$: dominated by $t \to 0$

2.) $|\Delta X| \ll \ell_s$: dominated by $t \to \infty$

In (2) we use the q-expansion around the cusp at ∞: Here ΔE is interpreted as a 1-loop diagram in the open string field theory on $YM(\mathcal{W}_{p+1})$. This is the sub-stringy D-geometry region: For more info see Kabat,Douglas,Pouliot, and Shenker.

Right now, we are focusing on (1). To understand this regime use the automorphic property to get an expansion around the cusp at $t = 0$:

$$f(t) \sim \sum_r c_r e^{-2\pi r/t}$$

We recognize in the t-integral the integral repn. of Bessel functions, which are just Green's functions: this is closed string exchange.

From the interaction energy you get the gravitational field as follows: Newton's law in higher dimensions for the gravitational potential $\Phi(r)$ generalizes as:

$$\Phi(r) = -\frac{GM}{r} \quad \text{in s} = 3 \quad \rightarrow \quad \Phi(r) = -\frac{\ell_{planck}^{D-2} T_p}{r^{d_T - 2}} \tag{5.1}$$

Here T_p is the energy/volume of a Dp-brane: From this Polchinski concluded his famous formula

$$T_p = const \frac{M_s^{p+1}}{g_s} \tag{5.2}$$

Moreover, since the gravitational potential is the perturbation $G_{00} = -1 + \Phi$ for weak fields, we find the metric outside a Dp-brane: [3]

$$\begin{aligned} G_{00} &\sim -\left(1 - const.g_s\left(\frac{\ell_s}{r}\right)^{d_T-2} + \cdots\right) \\ e^{-2\phi} &= e^{-2\phi_\infty}\left(1 - const.g_s\left(\frac{\ell_s}{r}\right)^{d_T-2} + \cdots\right) \end{aligned} \tag{5.3}$$

5.3.2. Relation to solutions of sugra equations

These asymptotics turn out to be exactly the asymptotics of nontrivial p-brane type solutions to the sugra equations.

In semiclassical field theory a nontrivial solution of the field equations corresponds to a quantum state through the coherent state formalism: $|\phi(x)\rangle$.

That is, there is a new sector of the spacetime stringy Hilbert space

Thus, the Dp branes can be considered as a microscopic description of new ~~objects~~ objects in the nonperturbative Hilbert space of weakly coupled string theory. They define states $|\Psi(\mathcal{W})\rangle_{SFT(\mathbb{R}^{1,s})}$.

Remarks.

1. It follows from (5.2) that at weak string coupling the Dbranes are very heavy. Heavy things make black holes, and it indeed turns out that D-branes provide models of black holes.
2. Nevertheless, (5.3) is only to be trusted in the regime where the low energy effective QFT is to be trusted, namely $r \geq \ell_s$. the gravitational field $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ as a perturbation on flat space is parametrically weak in g_s. This is important: we can trust leading order in string perturbation theory.
3. The above formulae are very definitely FALSE for the interesting case of $d_T \leq 2$. In this case the massless Green's function *grows* at long distances That is why $D7, D8, D9$ branes are much more confusing.
4. In the superstring context we will see that the Dbranes carry a conserved topological charge.

[3] Use the relation between Planck and string scales $\ell_{planck}^{D-2} = g_s^2 \ell_s^{D-2}$.

6. The Lagrangian for bulk + brane.

The system BULK + BRANE must have a single action.

For a single Dp-brane this action is believed to be:

$$L_{BULK} + \ell_s^{-p-1} \int_{\mathcal{W}} e^{-\phi} \sqrt{\det_{0\le\mu,\nu\le p}\left[G_{\mu\nu} + \ell_s^2(F_{\mu\nu} + B_{\mu\nu})\right]} + \cdots$$

Remarks:

1. DBI follows from T duality.
2. Very interesting feature is that only the combination $F_{\mu\nu} + B_{\mu\nu}$ enters.

A Polyakov path integral formulation of the sector $\mathcal{A}_{(\mathcal{B},\mathcal{B})}$ propagating in a background with A, B will involve:

$$\int dX(\sigma,\tau) \exp\left[-\ell_s^{-2}\int_\Sigma \langle \partial X, \bar\partial X\rangle + i\int_\Sigma X^*B + i\int_{\partial\Sigma} X^*(A)\right]$$

Preserving the spacetime 1-form gauge invariance requires introducing a nontrivial invariance of the worldvolume Maxwell theory:

$B \to B + d\Lambda$

$A \to A - \Lambda$

3. The nonabelian version of DBI is not completely clear.

6.0.1.SYM coupling and the UV status of YM

Set $B = 0$, assume the dilaton ϕ is constant, and expand DBI for small fieldstrengths F. Get:

$$\ell_s^{-p-1} g_s^{-1} \int_{\mathcal{W}} \left(\sqrt{\det G_{\mu\nu}} + \frac{\ell_s^4}{2}F^2 + \frac{\ell_s^8}{8}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{\ell_s^8}{24}(F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu}) + \cdots\right)$$

From the leading term we recover the formula for the tension of the D-brane. From the next term we obtain the relation between the Yang-Mills coupling on a Dp-brane and the bulk string parameters:

$$\frac{1}{g_{YM}^2} = \frac{1}{g_s}\left(\frac{1}{\ell_s}\right)^{3-p} \tag{6.1}$$

$p \le 3$: SYM is renormalizable

$p \ge 4$: g_{SYM}^2 has dimensions of a positive power of length. The SYM system is only an effective description at distances large compared to this scale.

This has important implications for Matrix theory - c.f. Dijkgraaf's lectures.

Exercise: KK Reduction $YM(\mathcal{B}_{p+1}) \to YM(\mathcal{B}_p)$ relates YM couplings via:

$$\frac{R'}{g_{YM,p+1}^2} = \frac{1}{g_{YM}^2}$$

Why is this compatible with (6.1) ?

T-duality section

7. Multi-brane sectors.

Since we have (extended) objects - D-branes - in our spacetime theory we can now consider situations where there are many of them- we should consider Fock spaces of D-branes.

7.0.1.Generalization of the CP construction

Consider first the two-brane subsector:

Consider two parallel Dp branes:

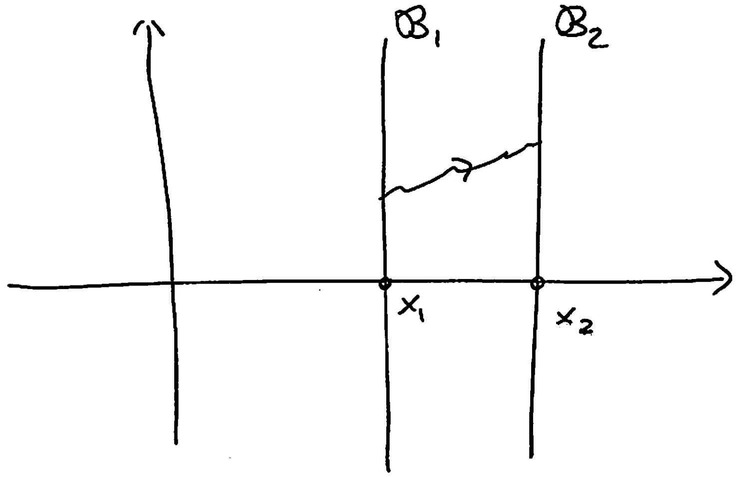


Fig. 16: Parallel branes with Chan-Paton spaces

The CFT statespace $\mathcal{A}_{(\mathcal{W}_1,\mathcal{W}_2)}$ was formulated above.

Now we must generalize the CP construction:

There is a natural generalization: to each coordinate plane $\mathcal{W}$ we attach a vector space $V_{\mathcal{W}}$ and now replace the CFT $\mathcal{A}_{(\mathcal{W}_1,\mathcal{W}_2)}$ by

$$\mathcal{A}_{(\mathcal{W}_1,\mathcal{W}_2)} \otimes Hom(V_{\mathcal{W}_1}, V_{\mathcal{W}_2})$$

$V_{\mathcal{W}}$ is called the Chan-Paton vector space and

$$\mathcal{W}_{p+1} \times V_{\mathcal{W}}$$

is the (trivial) Chan-Paton vector bundle.

Remark. In the theory of curved branes in superstring compactification the branes $\mathcal{B}$ wrap supersymmetric cycles in manifolds of special holonomy. The Chan-Paton vector space becomes replaced by a vector bundle $E \to \mathcal{B}$, and there are indications that the right way to think about this situation is in terms of a coherent sheaf on spacetime. In any case - in the next lecture we will consider instantons on Dp-branes, and this definitely involves nontrivial CP vector bundles.

7.0.2.Spacetime interpretation of the statespace for $\mathcal{B}_i \neq \mathcal{B}_j$

Consider two parallel Dp branes:

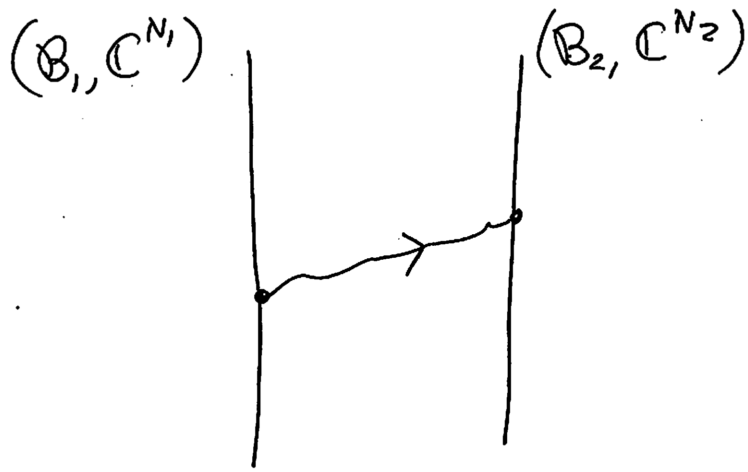


Fig. 17: Parallel branes with Chan-Paton spaces

The statespace $\mathcal{A}_{(\mathcal{B}_1,\mathcal{B}_2)}$ has a spectrum of particles:

$$L_0 = \ell_s^2(E^2 - \vec{p}^2) = \frac{(x_1 - x_2)^2}{\ell_s^2} + \mathbb{N} - 1$$

The first term on the RHS is there because strings have
Tension = energy/length = $1/\ell_s^2$
so, classically, a stretched string has minimal mass [4]

$$M \geq \frac{|\Delta x|}{\ell_s^2}$$

Moreover, these states transform under the $(\bar{N}_1, N_2)$ of the $U(N_1) \times U(N_2)$ gauge symmetry. The lowest mass (nontachyonic) states are spanned by (at level $\mathbb{N} = 1$):

$$\alpha_{-1}^{\mu}|k^\mu, X_{0,1}^a, X_{0,2}^a\rangle \otimes \lambda^m{}_i \qquad \mu = 0, \ldots, p$$
$$\alpha_{-1}^{a}|k^\mu, X_{0,1}^a, X_{0,2}^a\rangle \otimes \lambda^m{}_i \qquad a = p+1, \ldots, s$$

, $i = 1, \ldots, N_1$,$m = 1, \ldots, N_2$, $\lambda \in \mathrm{Hom}(\mathbb{C}^{N_1}, \mathbb{C}^{N_2})$ and consists of the reduction of a 10D massive vector in $(N_1, \bar{N}_2)$, so it is described by fields

$$(A_\mu(x))^m{}_i \qquad (\phi_a(x))^m{}_i \tag{7.1}$$

[4] Because of quantum corrections this reasoning is false for bosonic strings, but turns out to be correct for superstrings.

but where is x? Which brane is the field on?

Answer: If the energy is such that

$$E \ll \frac{|\Delta X|}{\ell_s^2}$$

then these states are not part of the low energy QFT description and the question doesn't make sense. But, if

$$\frac{|\Delta X|}{\ell_s^2} \ll E \ll \frac{1}{\ell_s}$$

then there is a good description in the low energy field theory. To find it notice that for

$$\Delta X \to 0$$

the CP spaces "merge" and the fields (7.1) become exactly the massless fields needed to fill out the $U(N_1+N_2)$ YM multiplet.

Now, within the $U(N_1+N_2)$ theory $YM(\mathcal{B}_p)$ suppose the scalars take vevs:

$$\langle 0|\phi_a|0\rangle_{YM(\mathcal{B}_p)} = \begin{pmatrix} \phi_a^1 1_{N_1} & 0 \\ 0 & \phi_a^2 1_{N_2} \end{pmatrix} \tag{7.2}$$

Then expanding around this vacuum $\phi_a \to \phi_a + \langle\phi_a\rangle$:

$$Tr_{\mathbb{C}^{N_1+N_2}}(D\phi)^2 = Tr_{\mathbb{C}^{N_1}}(D\phi)^2 + Tr_{\mathbb{C}^{N_2}}(D\phi)^2 + (\langle\phi_a^1 - \phi_a^2\rangle)^2 \sum_{i,m} |(A_\mu(x))^m{}_i|^2$$

Therefore, the vev in the theory $YM(\mathcal{B}_p)$ is identified with a relative position in the ambient spacetime:

$$(\phi_a^1 - \phi_a^2) = \ell_s^{-2}(\Delta X)^a$$

On distance scales large compared to (ΔX^a) but small compared to the string scale, the low energy dynamics is described by a Yang-Mills Higgs multiplet for $U(N_1+N_2)$ gauge theory, but the gauge symmetry is broken:

$$U(N_1+N_2) \to U(N_1) \times U(N_2)$$

by the displacement between the branes.

7.0.3. A family of D-brane sectors

Now we have opened up many possibilities: in the theory $YM(\mathcal{W}_{p+1})$ we could also consider vev's breaking symmetry like:

$$\langle 0|\phi_a|0\rangle_{YM(\mathcal{B}_p)} = \begin{pmatrix} \phi_a^1 1_{N_1} & 0 & 0 \\ 0 & \phi_a^2 1_{N_2} & 0 \\ 0 & 0 & \phi_a^3 1_{N_3} \end{pmatrix} \tag{7.3}$$

this has the spacetime interpretation of:

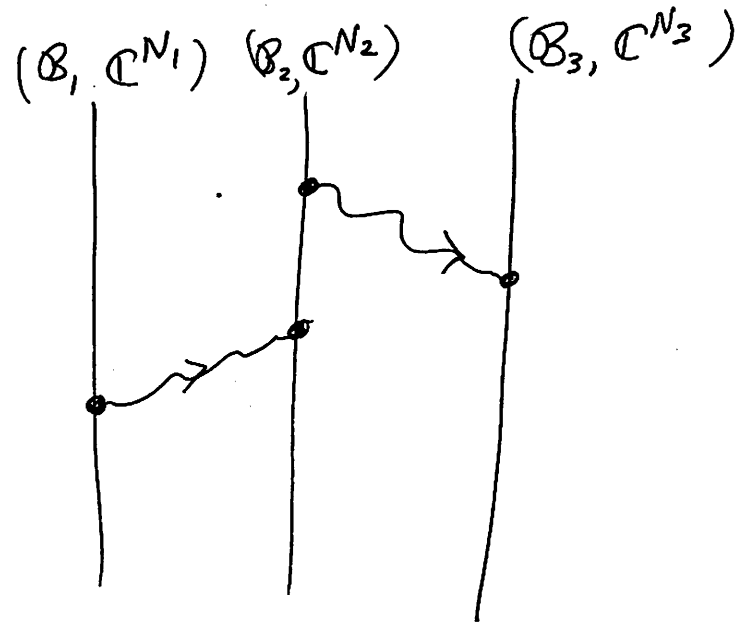


Fig. 18: 3 branes with CP factors.

Thus, we should consider a *family* of CFT statespaces over the symmetric product:

$$\begin{array}{c} \mathcal{A}_{p,N} \\ \downarrow \\ S^N \mathbb{R}^{d_T} \end{array}$$

What is the fiber?

$S^N \mathbb{R}^{d_T}$ is a stratified space. The strata are labelled by partitions of N. and over, say, the stratum

$$S^k_\nu \mathbb{R}^{d_T} = \{(\underbrace{x_1, \dots, x_1}_{n_1}, \underbrace{x_2, \dots, x_2}_{n_2}, \dots, \underbrace{x_k \dots, x_k}_{n_k})\} \cong S^k \mathbb{R}^{d_T}$$

The fiber is:

$$\mathcal{A}_{((\mathcal{B}_1,V_1),\dots,(\mathcal{B}_n,V_n))} = \oplus_{1\le i,j\le n} \mathcal{A}_{(\mathcal{B}_i,V_i,\mathcal{B}_j,V_j)} \otimes Hom(V_{\mathcal{B}_i}, V_{\mathcal{B}_j}) \quad , \quad \dim V_i = n_i$$

Thus: *each coordinate plane $\mathcal{B}$ defined by the Dirichlet boundary conditions comes equipped with a vector space $V_{\mathcal{B}}$. The CFT sectors needed to describe multi-Dbrane states are ordered pairs* $((\mathcal{B}, V), (\mathcal{B}', V'))$.

7.0.4.Connection to the moduli space of susy vacua

(To make the next point we really ought to be working in the context of superstring theory.)

The base space can be thought of as a moduli space in *two ways.*

In general - there is a beautiful and deep correspondence between phenomena on the worldsheet and phenomena in spacetime. This carries over nicely to branes - there is a correspondence between phenomena on the worldvolume and pheonomena in spacetime.

~~In the next lecture we will show the theory on a Dp brane is the reduction of 10D SYM theory with 16 supercharges.~~

In the present case we have:

S: Parallel Dp branes have no force between them. They are exact eigenstates of the spacetime string field theory Hamiltonian [5]

W: The low energy theory on the brane is the reduction of 10D SYM with 16 supercharges, and has a moduli space of supersymmetric vacua.

The supersymmetric moduli space of vacua of the brane theory is found by minimizing the potential energy:

$$\mathcal{M} = \{\langle\phi\rangle : V = \sum_{a<b} \text{tr } [\phi_a, \phi_b]^2 = 0\}/U(N)$$

Clearly:

$$\mathcal{M} = S^N \mathbb{R}^{d_T} \equiv (\mathbb{R}^{d_T} \times \cdots \times \mathbb{R}^{d_T})/S_N$$

The singular orbifold subvarieties correspond to the phenomena of:

W) Vacua with enhanced gauge symmetry.

S) Coincident Dbranes.

The moduli space of supersymmetric vacua is identified with a configuration space of Dbranes in spacetime.

Remark. Thus, the ϕ_a are some kind of noncommutative coordinates on spacetime. This observation has been much further developed in matrix theory.

8. T-duality and D-branes

So far we have taken spacetime $\mathcal{S} = \mathbb{R}^{1,s}$.

Something dramatically different happens when we compactify. We will simply take $\mathcal{S} = \mathbb{R}^{1,s-d} \times T^d$, with T^d a flat torus.

[5] whatever that is ..., but using supersymmetry we can make such an assertion.

8.0.1. T duality of the massless scalar

Recall for a closed string:

W: Electric-magnetic duality

S: T-duality

$$\phi = \phi_R(t-x) + \phi_L(t+x)$$
$$d\tilde{\phi} = *F = *d\phi$$
$$\tilde{\phi} = -\phi_R(t-x) + \phi_L(t+x)$$

So, from the oscillator expansion:

$$\begin{aligned} \varphi &= \varphi_0 + \sqrt{2}\,\frac{n\ell_s^2}{R}\,t + \sqrt{2}\,mR\sigma + \phi_{\rm osc}. \\ &= \varphi_0 + \frac{1}{\sqrt{2}}\,(\frac{n\ell_s^2}{R} + mR)(t+\sigma) + \frac{1}{\sqrt{2}}\,(\frac{n\ell_s^2}{R} - mR)(t-\sigma) + \phi_{\rm osc}. \end{aligned}$$

T-duality takes

$$R \to R' = \ell_s^2/R \tag{8.1}$$

This is an exact duality of perturbative closed string theory. Now, what happens when we include D-branes?

8.0.2. D-brane with a compact transverse direction

Let us consider a Dbrane with a compact transverse circle:

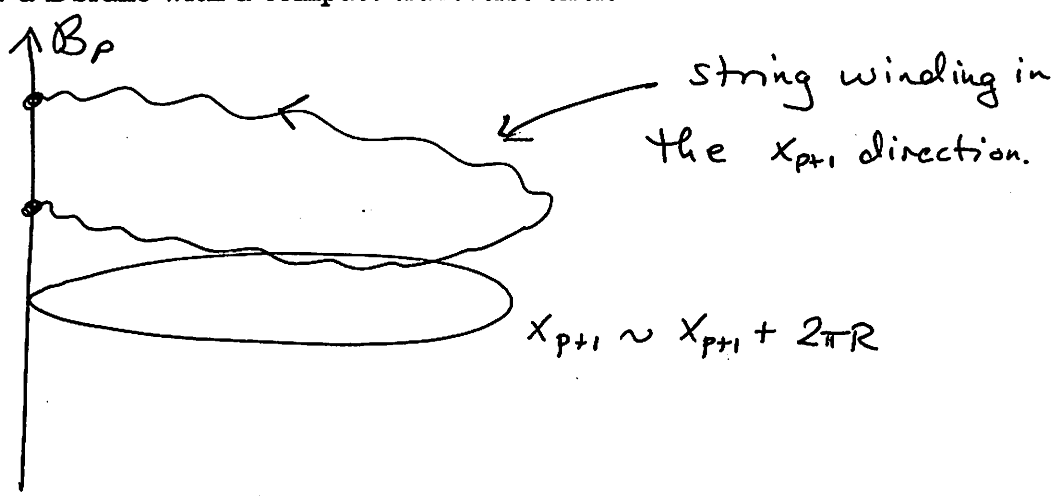


Fig. 19: Dbrane with a compact transverse direction.

Now the sectors of the CFT have a new conserved quantity: The winding number:

$$\mathcal{A}_{(\mathcal{B},\mathcal{B})} = \oplus_{w\in\mathbb{Z}}\mathcal{A}^{(w)}_{(\mathcal{B},\mathcal{B})}$$

with oscillator expansion:

$$X^{p+1} = X_0^{p+1} + wR\sigma + i\sum_{m\in\mathbb{Z}-0}\frac{\alpha_m}{m}(z^{-m}+\bar{z}^{-m})$$

We can write the spectrum immediately since the strings have known tension:

$$\ell_s^2(E^2-\vec{p}^2) = \frac{(2\pi Rw)^2}{\ell_s^2} + \mathbb{N} - 1$$

Let us focus on the $N=1$ states. These are spanned by $A^{(w)}_\mu(x)$. These are *massive gauge bosons.* We have an infinite collection of massive gauge bosons of mass $\frac{2\pi R|w|}{\ell_s^2}$

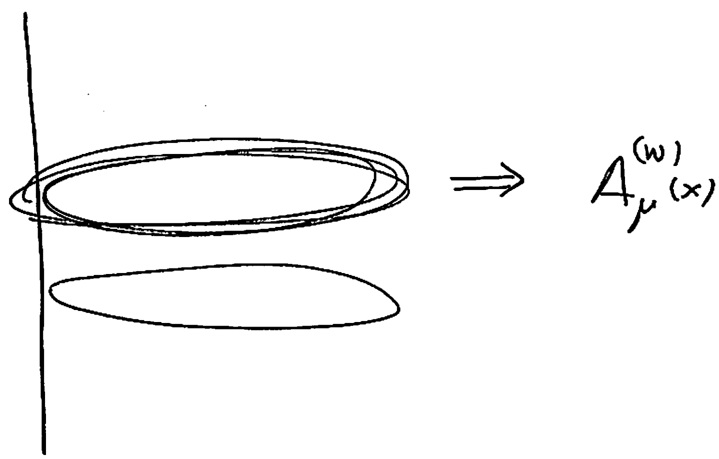


Fig. 20: Winding string gives a massive gauge boson on the Dbrane.

This tower of particles looks like a Kaluza-Klein tower of vector bosons and scalars, so consider the KK reduction of $YM(\mathcal{W}_{p+1}\times T^1_{R'})$. Reduction amounts to fourier decomposition:

$$A(x^\mu,y) = \sum_{w\in\mathbb{Z}} e^{2\pi iwy/R'}A^{(w)}(x) \qquad \mu=0,\dots,p$$

$$\phi_a(x^\mu,y) = \sum_{w\in\mathbb{Z}} e^{2\pi iwy/R'}A^{(w)}(x) \qquad a=p+1,\dots,s$$

Each gauge boson is associated with a gauge group $Map[\mathbb{R}^{1,p} \to U(N)]$. If we take all these together [6] this is just the affinization of the gauge group

$$\hat{\mathcal{G}} = Map[T^1_{R'} \to Map[\mathbb{R}^p \to U(N)]] \cong Map[\mathbb{R}^p \times T^1_{R'} \to U(N)]$$

with no central extension.

The KK reduction gives a tower of particles of mass w/R'. Therefore

$$R' = \ell_s^2/R \tag{8.2}$$

We recover the above familiar rule of T-dualtiy.

Claim: This continues to hold for the interactions.

Proof: Use EM duality on the worldsheet with D and N boundary conditions.

Conclusion: We learn that T-duality connects all Dp branes of different p, as long as there are compact dimensions to dualize:

$$\begin{aligned} \mathcal{T}_\perp : &\quad Dp \to D(p+1) \\ \mathcal{T}_\parallel : &\quad D(p+1) \to Dp \end{aligned} \tag{8.3}$$

Remark: Which description we use depends on scales. When $R \ll \ell_s$ there are many light states and it is more appropriate to describe the system in terms of a $(p+1)$-dimensional gauge theory. When $R \gg \ell_s$ we should use $YM(\mathcal{B}_p)$.

8.0.3. Wilson lines and positions

Now let us consider a situation with several branes at positions

$$\theta_1 = y_1/R' \qquad \cdots \qquad \theta_k = y_k/R'$$

as in fig.21 below.

The mass of the gauge bosons associated with winding strings is now:

$$mass \sim |\theta_i - \theta_j| + 2\pi R w/\ell_s^2$$

This simply corresponds to KK reduction of a gauge theory in $\mathbb{R}^{1,p+1}$ along y with a flat gauge field: $A^0_{p+1}dy$ s.t.:

$$\text{Pexp} \oint A^0_{p+1}dy \sim \exp\left[2\pi i Diag\{\theta_1 1_{N_1}, \ldots, \theta_k 1_{N_k}\}\right]$$

$$Tr(D_{A^0}A)^2 = \sum |DA^{ii}|^2 + \sum(\theta_i - \theta_j)^2 |(A_\mu)^i{}_j|^2$$

Under the duality map $\mathcal{T}_\parallel$ in (8.3) the eigenvalues of the Wilson line of the $(p+1)$-brane encode the positions of the p-branes. Once again, moduli of groundstates in a gauge theory are identified with spacetime positions.

[6] and ignore all other string field theory gauge invariances...

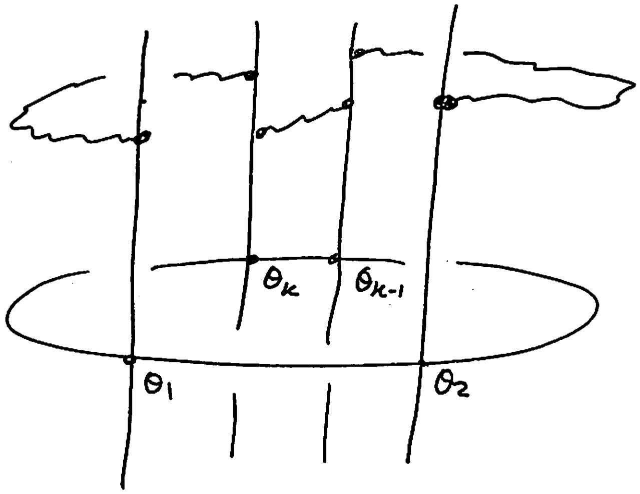


Fig. 21: Several Dbranes with a compact transverse direction.

D-Branes 101b

The bosonic strings and their Dbranes discussed above are not really consistent. We need supersymmetry. From now on we work in 10 dimensional superstrings. $s = 9$.

9. Superconformal field theory with a boundary

Action:

$$S = \frac{1}{4\pi}\int_\Sigma \ell_s^{-2}\partial X^\mu \bar\partial X_\mu + \psi^\mu \bar\partial \psi_\mu + \tilde\psi^\mu \partial \tilde\psi_\mu$$

We start with $\Sigma =$ the strip.

Boundary conditions for bosons are chosen as D or N as before.

For the ws fermions $\psi, \tilde\psi$, since we have first order systems the only way to cancel the boundary terms is to relate ψ to $\tilde\psi$. Two inequivalent boundary conditions on the fermions are:

$$R : \psi^\mu(\sigma^1 = 0) = \tilde\psi^\mu(\sigma^1 = 0)$$
$$\psi^\mu(\sigma^1 = \pi) = \tilde\psi^\mu(\sigma^1 = \pi)$$
$$NS : \psi^\mu(\sigma^1 = 0) = -\tilde\psi^\mu(\sigma^1 = 0)$$
$$\psi^\mu(\sigma^1 = \pi) = \tilde\psi^\mu(\sigma^1 = \pi)$$

Since modular transformations mix spin structures a consistent superconformal field theory requires introducing both boundary conditions so that the CFT statespace has a $\mathbb{Z}_2$-grading:

$$\mathcal{A} = \mathcal{A}_{NS} \oplus \mathcal{A}_R$$

In addition to the choices of N, D at the two boundaries.

9.1. Oscillators and quantization

Quantization of the oscillator expansions associated with the above bc's gives a Heisenberg and a Clifford algebra:

$$\begin{aligned}[][\alpha_r^\mu, \alpha_{r'}^\nu] &= r\delta_{r+r',0}\eta^{\nu\mu} \\ \{\psi_r^\mu, \psi_{r'}^\nu\} &= \delta_{r+r',0}\eta^{\mu\nu}\end{aligned} \tag{9.1}$$

Here r, r' are in $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, which depends on the 8 possible boundary conditions:

	NS	R
NN	$\alpha_n, \psi_{n+1/2}$	α_n, ψ_n
DD	$\alpha_n, \psi_{n+1/2}$	α_n, ψ_n
$(DN), (ND)$	$\alpha_{n+1/2}, \psi_n$	$\alpha_{n+1/2}, \psi_{n+1/2}$

Table 1: Modings for the 8 boundary conditions. n is an integer.

The main new ingredient is that the level zero states $\mathbb{N}=0$ must form a representation of the NSR fermions with moding zero:

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$$

We choose an irrep spanned by states $|\alpha\rangle$.

For (N, N) on all coordinates:

$$Span\{|\alpha\rangle\} = \mathbf{16}_+ \oplus \mathbf{16}_- \cong \mathbb{R}^{32}$$

We represent these algebras via a Fock $\otimes$ Grassmann representation (graded symmetric algebra on the positive frequency modes) so that

$$\begin{aligned}\mathcal{A}_{NS} &= \oint dp \mathcal{F}_p^{X,\psi} \\ \mathcal{A}_R &= \oint dp \mathcal{F}_p^{X,\psi} \otimes (16_+ \oplus 16_-)\end{aligned}$$

9.2. CFT Statespaces + Spectrum

For each coordinate direction (X^μ, ψ^μ) we have:

9.2.1.NS sector

NS groundstate energies:

$$DD, NN: \qquad (\alpha_n, \psi_{n+1/2}) \qquad -\frac{1}{24}-\frac{1}{48}=-\frac{3}{2}\frac{1}{24}=-\frac{1}{16}$$

$$DN: \qquad (\alpha_{n+1/2}, \psi_n) \qquad +\frac{1}{48}+\frac{1}{24}=+\frac{3}{2}\frac{1}{24}=+\frac{1}{16}$$

Note the integer-moded DN fermions - this will play an important role.

$$NN: L_0=-\frac{1}{16}+\ell_s^2p^2+\mathbb{N}$$

$$DD: L_0=-\frac{1}{16}+\frac{(x_2-x_1)^2}{\ell_s^2}+\mathbb{N}$$

$$DN: L_0=+\frac{1}{16}+\mathbb{N}$$

Adding the groundstate energies of all the coordinates:

$$E=\frac{1}{16}(\nu_{ND}-\nu_{DD}-\nu_{NN})=-\frac{1}{2}+\frac{\nu_{DN}}{8}$$

$\nu=$ number of coordinate directions of given type.

9.2.2.R-sector

Groundstate energy always zero.

Spectrum:

$$\begin{aligned} NN: &\quad L_0=\ell_s^2p^2+\mathbb{N} \\ DD: &\quad L_0=\frac{(x_2-x_1)^2}{\ell_s^2}+\mathbb{N} \\ DN: &\quad L_0=\mathbb{N} \end{aligned}$$

10. Superstrings

These have many formulations. We will use the covariant NSR formalism. For more detail see CJP, Polchinski 3.1,3.2

10.1. Open superstrings

As before we will start with the case (N, N) on all coordinates (X^μ, ψ^μ).

Want: Hilbert space is unitary representation of the minimal superpoincare algebra $\mathcal{SP}(1,9)$.

10.1.1.GSO projection

The infinite dimensional Clifford algebra has a chirality operator:

$$
\begin{aligned}
(-1)^F &= (-1)\prod_{r>0}(-1)^{\psi_{-r}\psi_r} && NS \\
&= \psi_0^0\psi_0^1\cdots\psi_0^9\prod_{r>0}(-1)^{\psi_{-r}\psi_r} && R
\end{aligned}
$$

Fact: The chirally projected theories:

$$\mathcal{A}^\pm = \mathcal{A}_{NS}^+ \oplus \mathcal{A}_R^\pm$$

are consistent CFT's. Note $\mathcal{A}_R^\pm|_{\mathbb{N}=0} \cong 16_\mp$.

Moreover, they admit a conserved $\Delta = 1$ local "spacetime supersymmetry current"

$$\epsilon^\alpha(j_\alpha(z) + \tilde{j}_\alpha(\bar{z})$$

where $\epsilon \in \mathbf{16}_\pm$.

10.1.2.BRST cohomology

The on-shell statespace defined by the Lie algebra cohomology of the superVirasoro algebra:

$$\mathcal{H} \equiv H_d^*(\mathcal{A}_{NS}^+ \oplus \mathcal{A}_R^+)$$

is a completely reducible unitary reprsentation of $\mathcal{SP}(1,9)$.

At $\mathbb{N} = 0$ the states are massless:

$\mathcal{A}_{NS}$: $\epsilon_\mu\psi^\mu_{-1/2}|p\rangle$

Kerd: $\epsilon\cdot p = 0$, $\epsilon \sim \epsilon + xp$, as before

$\mathcal{A}_R$ On the states $|\alpha\rangle$

d, the Dirac-Ramond operator, reduces to the dirac operator:

kerd:

$$p_\mu \psi_0^\mu |p,s\rangle = 0 \leftrightarrow \not{D} s(x) = 0$$

The representation of the massless little group $Spin8$ is:

$$8_v \oplus 8_-$$

Including the Chan-Paton construction the massless particle spectrum as a representation of $Spin(1,9) \times U(N)$ is

$$Ind(8_v \oplus 8_+) \otimes u(N)$$

forming an on-shell VM representation of the algebra $\mathcal{SP}_{16}(1,9)$ with 16 real supercharges.

Field repn: (A_M, χ) where the gaugino χ is in the 16_-.

Low energy action:

$$S = \frac{1}{g_{YM}^2} \int Tr(F * F + \bar{\chi} \not{D} \chi)$$

10.2. Closed superstrings

Now an interesting new feature appears: The statespace is

$$\mathcal{A}_{closed} \equiv (\mathcal{A}_{NS}^{+} \oplus \mathcal{A}_{R}^{\pm}) \otimes (\tilde{\mathcal{A}}_{NS}^{+} \oplus \tilde{\mathcal{A}}_{R}^{+})$$

The choice $\pm$ defines the IIB, IIA theory.

Spacetime interpretation: (NS,NS) $\Rightarrow G, B, \phi$, as before.

A major new factor is the (R,R) sector, describing spacetime bosons. Notice that these states and fields are of the type $(spinor) \otimes (spinor)$ and are therefore associated with differential forms. Indeed, recall the decompositions for MW spinors of $Spin(1,9)$:

$$\begin{aligned} \mathbf{16}_- \otimes \mathbf{16}_+ &= \left[\Omega^{even}(\mathcal{S})\right]^+ \\ \mathbf{16}_+ \otimes \mathbf{16}_+ &= \left[\Omega^{odd}(\mathcal{S})\right]^+ \end{aligned} \tag{10.1}$$

The superscript $+$ on the RHS means Hodge self-dual.

In indices, the relevant part of the string field is:

$$\Psi = \int dp \sum_k \jmath_\alpha(z) \otimes \tilde{\jmath}_\beta(\bar{z}) (\Gamma^{\mu_1 \cdots \mu_k} C)^{\alpha\beta} e^{ip\cdot X} G_{\mu_1 \cdots \mu_k}(p)$$

Thus we have fields:

$$IIA: \quad G \in [\Omega^0 \oplus \Omega^2 \oplus \Omega^4 \oplus \Omega^6 \oplus \Omega^8 \oplus \Omega^{10}]^+$$
$$IIB: \quad G \in [\Omega^1 \oplus \Omega^3 \oplus \Omega^5 \oplus \Omega^7 \oplus \Omega^9]^+$$

How shall we interpret them?

a.) The physical state conditions come from the Dirac-operator on left and right-moving spaces. Under the isomorphism (10.1) the dirac operator becomes d so the onshell conditions from $ker d$ are:

$$dG = 0$$
$$d * G = 0$$

b.) The G's are already *gauge invariant* under $\Psi \to \Psi + d\Lambda$

Conclusion: The dynamical fields are given by potentials defined locally by $G^{(p+2)} = dC^{(p+1)}$*:*

$$IIA: \quad C^{(1)} \in \Omega^1 \qquad C^{(3)} \in \Omega^3$$
$$IIB: \quad C^{(0)} \in \Omega^0 \qquad C^{(2)} \in \Omega^2 \qquad C^{(4)} \in \Omega^4$$

That is, type II sugra contains generalized Maxwell theories of forms.

Remarks

1. It is convenient to form the total potential $C = \sum_p C^{(p+1)}$, note that $C = \kappa * C$.
2. The low energy lagrangian involves:

$$\int dC^{(p+1)} \wedge * dC^{(p+1)}$$

3. The Ω^0, Ω^{10} contributions above are interesting but confusing. We ignore them here.

10.3. Generalized Maxwell theories, EM duality, Hodge duality, and all that

Thus, type II sugra involves generalized form theories:

$$G^{(p+2)} = dC^{(p+1)}$$
$$C^{(p+1)} \to C^{(p+1)} + d\Lambda^{(p)}$$

The sources for Bianchi/EOM are respectively the magnetic/electric currents:

$$dG^{(p+2)} = J_m \in \Omega^{p+3}$$
$$*d * G^{(p+2)} = J_e \in \Omega^{p+1}$$

(in the presence of localized sources $G^{(p+2)}, *G^{(p+2)}$ are "currents" in the mathematical sense.)

In particular, a p-brane soliton localized on $\mathcal{W}_p$ which is electrically charged defines a source which is a δ-function representative of the Poincare dual cohomology class $\eta(\mathcal{W} \hookrightarrow \mathcal{S})$:

$$d * G^{(p+2)} = q_e \eta(\mathcal{W} \hookrightarrow \mathcal{S})$$

The charge $q_e$is measured by the integral in the linking sphere in the normal bundle:

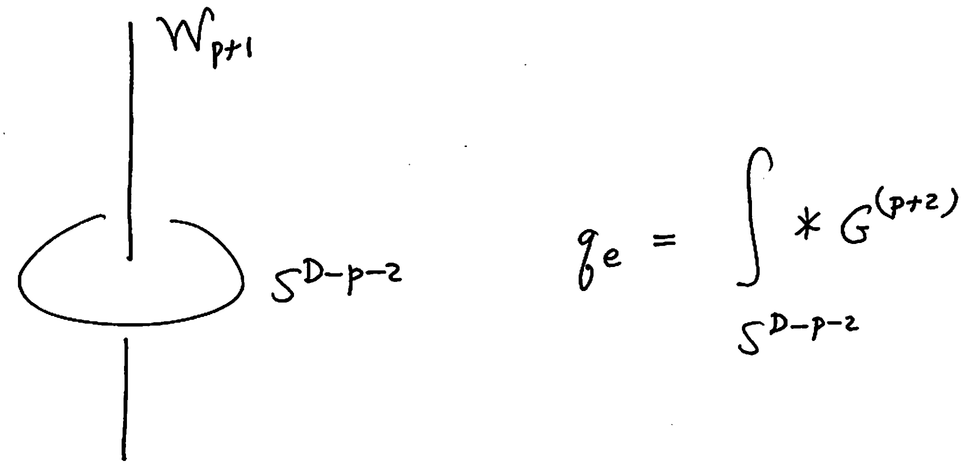


Fig. 22: Linking sphere for electric brane

In a path-integral for the field $C^{(p+1)}$ the electric source is introduced via the factor:

$$\exp\left[i \int_{\mathcal{W}_{p+1}} C^{(p+1)}\right]$$

Put differently, the charge is a rank-p antisymmetric tensor under the spatial isometry group:

$$Z_{M_1 \ldots M_p} = \int_{fixed\ time} d^s \vec{x} (J_e)_{0 M_1 \cdots M_p}$$

A $D-p-4$ brane can be magnetically charged under $C^{(p+1)}$:

$$dG^{(p+2)} = \eta(\mathcal{W}_{D-p-3} \hookrightarrow \mathcal{S})$$

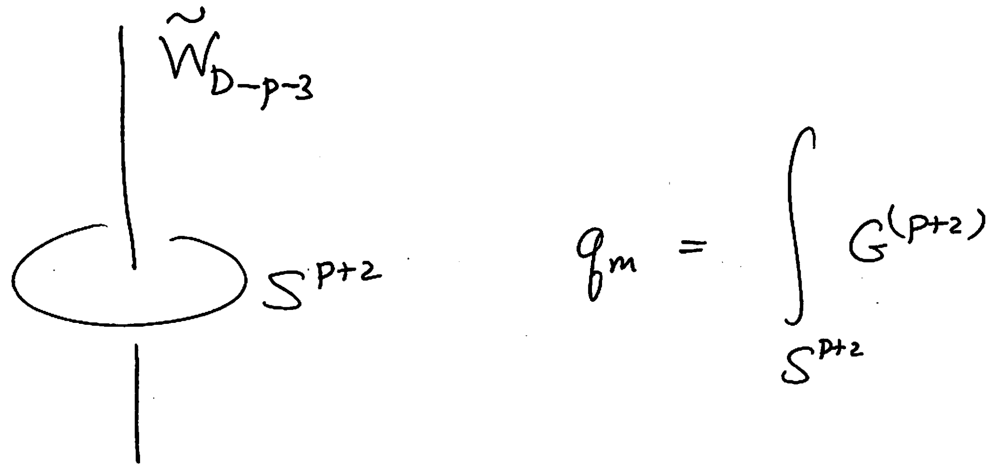


Fig. 23: Linking sphere for magnetic brane

A magnetic source cannot be introduced into the $C^{(p+1)}$ path integral in a local way. Using the Hodge * we can define a duality:

$$dC^{(p+1)} = G^{(p+2)} \overset{*}{\leftrightarrow} * G^{(p+2)} = \tilde{G}^{(D-p-1)} = d\tilde{C}^{(D-p-3)}$$

Remarks

1. The nonabelian generalization of all this is extremely interesting. M-theory promises that it exists.
2. Type II sugra solitons with p-form charges have been constructed. Note, since the vertex operator of RR multiplies a *fieldstrength* $G \sim dC$ the coupling to C vanishes at $p = 0$ and hence: *All perturbative string states are RR neutral.*
3. the intersection numbers of the linking spheres is the symplectic inner product of the electric/magnetic charges.

Summary: Type II SUGRA RR Forms and their branes:

	$C^{(1)}$	$C^{(7)}$
$\mathcal{B}_0$	E	M
$\mathcal{B}_6$	M	E

Type IIA

	$C^{(3)}$	$C^{(5)}$
$\mathcal{B}_2$	E	M
$\mathcal{B}_4$	M	E

Type IIA

	$C^{(2)}$	$C^{(6)}$
$\mathcal{B}_1$	E	M
$\mathcal{B}_5$	M	E

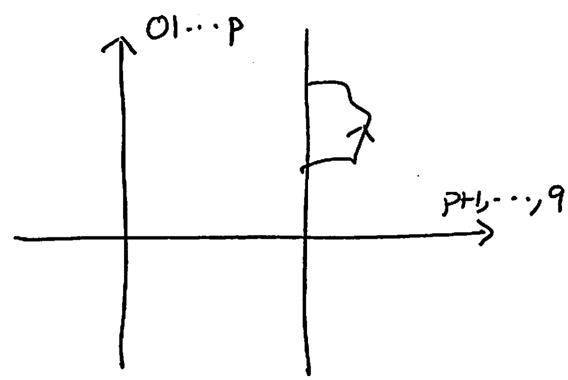
Type IIA

$\mathcal{B}_3$ is selfdual under $C^{(4)}$.

11. D-branes as BPS states

We now consider the effect of D,N boundary conditions. We focus on $\mathcal{A}^+_{\mathcal{B},\mathcal{B}}$ which has boundary conditions:

	0,1,..., p	p+1,..., 9
$\sigma = 0$	N	D
$\sigma = \pi$	N	D



11.1. Global symmetries of $\mathcal{A}^+_{\mathcal{B},\mathcal{B}}$

The presence of the D-brane breaks the Lorentz symmetry:

$$Spin(1,9) \supset Spin(1,p) \times Spin(d_T) \qquad (11.1)$$

but dimensional reduction preserves the 16 supercharges of the 10D SYM. Thus we have the family of theories with 16 supercharges:

$$[d=10, \mathcal{N}=(1,0)] \rightarrow [d=6, \mathcal{N}=(1,1)] \rightarrow [d=4, \mathcal{N}=4] \rightarrow [d=2, \mathcal{N}=(8,8)]$$

We denote the extended superalgebras in these dimensions by $\mathcal{SP}_{16}(1,p)$. The supersymmetry charges transform under the Lorentz and "R-symmetry" group $Spin(d_T)$ in the representation S^+ coming from the $\mathbf{16}_+$ In particular we have [7]

$$\begin{aligned} S^+ &\cong (4;2,1)_{\mathbb{R}} \oplus (\bar{4};1,2)_{\mathbb{R}} \qquad && Spin(1,5)\times Spin(4)_{6789} \\ &\cong (2,1;4)\oplus(1,2;\bar{4}) && Spin(1,3)\times Spin(6)_{456789} \end{aligned}$$

11.2. Massless modes on the brane

Massless modes:

W	S	
$\psi^{\mu}_{-1/2}\vert k,X_0\rangle_{NS}$	$A_\mu(x)$	$A\in\mathcal{A}(\mathcal{W}_{p+1};u(N))$
$\psi^{a}_{-1/2}\vert k,X_0\rangle_{NS}$	$\phi_a(x)$	$\phi_{p+1,\dots,9}\in\Omega^0(\mathcal{W}_{p+1};u(N))$
$\vert k,X_0,s\rangle_R$	$\chi_{\alpha A}(x)$	$\chi\in\Gamma[S^-\otimes u(N)]$

This is just the dimensional reduction of 10D SYM giving a susy Yang-Mills-Higgs with Lagrangian:

$$\frac{1}{g^2_{YM}}\int d^{p+1}\xi\left[\mathrm{tr}\ F^2+\mathrm{tr}\ (D\phi)^2+\mathrm{tr}\ [\phi,\phi]^2+\mathrm{tr}\ \bar{\chi}D\chi+\mathcal{O}((E\ell_s)^4,g_s)\right]+$$

11.3. Broken Supersymmetries in the bulk theory

Recall from the previous lecture that D-branes are objects defining states in the Hilbert space of the ambient 10-dimensional spacetime theory. Let us call these states $|\Psi(\mathcal{B},V)\rangle$.

Notice: The brane breaks translation invariance. Throw a ball against a wall: The brane breaks translation invariance and P is not conserved. But:

$$\{Q,Q\}\sim P+\cdots$$

hence some supersymmetries are broken. What is much more surprising: some supersymmetries are preserved!

The bulk type II theory has 32 supersymmetries:

$$\epsilon^\alpha Q_\alpha+\tilde{\epsilon}^\alpha\tilde{Q}_\alpha$$

are conserved charges for arbitrary $(\epsilon,\tilde{\epsilon})\in\mathbf{16}_\pm\oplus\mathbf{16}_+$

But the bulk interacts with the brane, and the brane only has 16 conserved supersymmetries.

[7] The subscript $\mathbb{R}$ means symplectic MW condition.

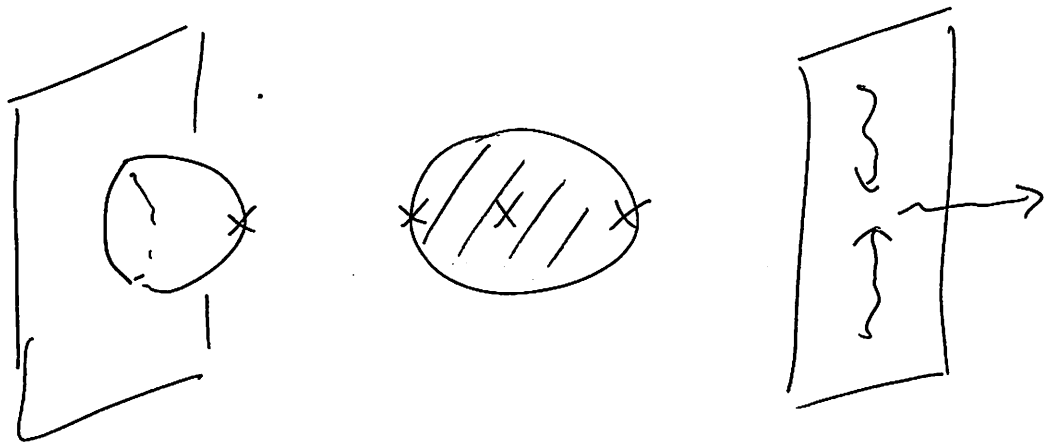


Fig. 24: Three pictures representing the interactions between D-brane and bulk.

Theorem In the sector $\mathcal{A}_{(\mathcal{B},\mathcal{B})}$ the conserved spacetime supercurrent is

$$(\epsilon, j)(z) + (\tilde{\epsilon}, \tilde{j})(\bar{z})$$

where $\tilde{\epsilon}$ and ϵ are related by:

$$\boxed{\tilde{\epsilon} = \Gamma^{01\cdots p}\epsilon} \tag{11.2}$$

Proof: If all boundary conditions are (N,N) then the conserved supercurrent is just $\epsilon^\alpha(j_\alpha(z)+\tilde{j}_\alpha(\bar{z})$. The reason is that the conservation of the current requires that no current can "flow off the end of the interval." In the upper half plane picture this means

$$\Big[(\epsilon, j)(z) - (\tilde{\epsilon}, \tilde{j})(\bar{z})\Big]_{z=\bar{z}} = 0$$

Now, in the R sector the ws fermions satisfy: $\psi|_{z=\bar{z}} = \tilde{\psi}|_{z=\bar{z}}$. hence $j = \tilde{j}$, and therefore only spinors with $\tilde{\epsilon} = \epsilon$ give conserved currents. Now, T-duality allows us to map conserved supercharges for differing values of p. Each T-duality transformation requires a parity flip in each coordinate $(\tilde{X}^\mu, \tilde{\psi}^\mu$, preserving X. This produces (11.2). ♠

Interpretation: For supersymmetry transformations satisfying (11.2),

$$(\epsilon^\alpha Q_\alpha + \tilde{\epsilon}^\alpha \tilde{Q}_\alpha)|\Psi(\mathcal{B}, V)\rangle = 0$$

$\in$ S.F.T($\mathbb{R}^{1,9}$)

Thus, the Dbrane is a state which preserves 1/2 *the supersymmetries of the ambient type II theory: Therefore, it is a BPS state in that theory!*

12. *Dp*-branes and supergravity solutions

The result of the previous section has an important consequence. BPS states carry a charge and satisfy the Bogomolnyi bound. That is how it is consistent to have states with positive energy yet annihilated by Q.

Since the global symmetry is (11.1) the algebra must be:

$$\{Q,Q\} = C\Gamma^M P_M + C\Gamma^{M_1\cdots M_p} Z_{M_1\cdots M_p}$$

Now, recall the discussion above of charges in generalized Maxwell theories. p-form charges arise from electrically charged p-branes. The only $(p+1)$-form charge available is the RR charge.

Dp branes are the microscopic description of the supergravity solutions with RR charge.

12.1. Unbroken supersymmetries

The crucial condition (11.2) can also be derived from the macroscopic viewpoint.

The extremal supergravity soliton equations can be derived from the *Killing spinor equations*:

$$\begin{aligned} \delta_\epsilon \lambda &= (\ldots)\epsilon = 0 \\ \delta_\epsilon \psi_M &= \mathcal{D}_M \epsilon + (\ldots.)\epsilon = 0 \end{aligned} \tag{12.1}$$

Under the condition (11.2) these equations lead to first order equations on the fields generalizing the self-duality equation $F = *F$ to supergravity.

Remarks:

1. *D*-branes were first constructed as an oddity in T-duality. String/U-duality demanded the existence of states with RR charge. Polchinski then realized that Dbranes carry RR charge. The simplicity of this description, and the fact that it gives a consistent microscopic description of the RR solutions then lead to an explosion of activity.
2. **Warning:** There are other p-branes branes in supergravity, not carrying RR charge, and they play an important role.

12.2. Measuring the mass

The mass of the *Dp*-branes can be calculated just as before by calculating the interaction energy from:

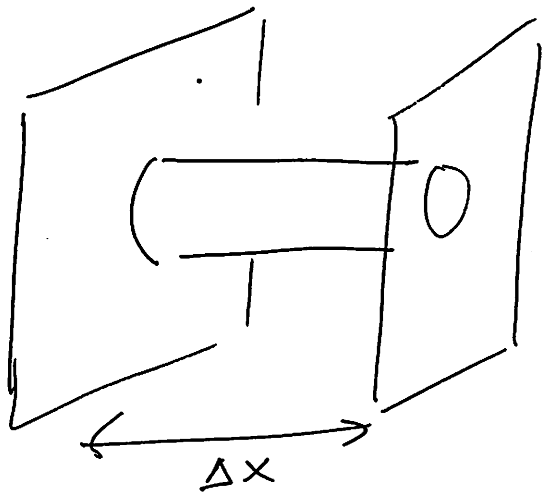


Fig. 25: tube between branes.

One finds the interaction energy to be:

$$\Delta\mathcal{E} \sim \int_0^\infty \frac{dt}{t} t^{-(p+1)/2} e^{-t(\Delta X)^2/\ell_s^2} \left(\sum_{\mathcal{A}_{NS}} e^{-tm^2} - \sum_{\mathcal{A}_R} e^{-tm^2} \right)$$

Actually, by supersymmetry this vanishes. So we have justified the statement in the previous lecture

Two Dp branes of like charge feel no force. Hence the Dp brane of charge $N_1 + N_2$ is a boundstate at threshold.

Nevertheless, considering the separate terms we can measure the mass as in the bosonic case:

$$T_p = const. \frac{\ell_s^{-p-1}}{g_s}$$

12.3. BULK + BRANE lagrangian

Once again we have the system defined by:

$$L_{IIA\ SUGRA} + \ell_s^{-p-1} \int_{\mathcal{W}} e^{-\phi} \sqrt{\det_{0\le\mu,\nu\le p} \left[G_{\mu\nu} + \ell_s^2 (F_{\mu\nu} + B_{\mu\nu}) \right]} + CORRECTION$$

Now there is a very important correction discussed in the next section, which enters at leading order in the low-energy expansion.

13. Instantons as solitons in $YM(\mathcal{B}_{p+4})$

Now we want to start asking: what happens when the SYM is not in its groundstate? In general the answer is very complicated, but for certain states we can say something - these are the solitonic states of the Yang-Mills theory defined by mathematical instantons.

We consider a $D(p+4)$-brane whose worldvolume includes the directions $\mathbb{R}^4_{6789}$. The low energy theory is a Yang-Mills theory $YM(\mathcal{B}_{p+4})$, and we can consider a Chan-Paton bundle

$$E \to \mathcal{W}_{p+5}$$

which is topologically nontrivial (with L^2 conditions at ∞).

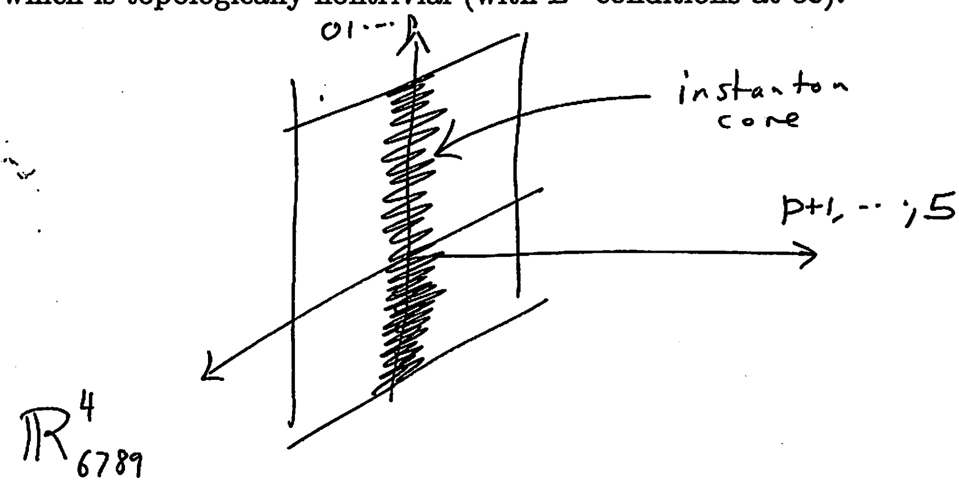


Fig. 26: A Yang-Mills instanton in the theory $YM(\mathcal{B}_{p+4})$.

Choose a coordinate subspace $\mathcal{B}_{p+4} = \mathcal{B}_p \times \mathbb{R}^4_{6789}$.

The Yang-Mills instantons on $\mathbb{R}^4_{6789}$ which are translationally invariant in the $0, 1, \ldots, p$-directions define p-dimensional solitons in $YM(\mathcal{B}_{p+4}, E)$. The energy density is proportional to the action density for the 4d yang-mills theory on $\mathbb{R}^4_{6789}$.

Note - the mathematical instanton defines a p-brane physical object in the SYM theory $YM(\mathcal{B}_{p+4})$

$p = 0$ particle

$p = 1$ string

$p = 2$ membrane

Let us call this state $|\Psi(A)\rangle_{YM(\mathcal{B}_{p+4})}$.

The solitonic state $|\Psi(A)\rangle_{YM(\mathcal{B}_{p+4})}$ is a BPS state in the SYM theory.

Proof:

The instanton satisfies the equations:

$$F_{MN} = 0 \qquad M, N \notin \{6,7,8,9\}$$

$$F = *F \qquad M, N \in \{6,7,8,9\}$$

Consider the supersymmetry variation for a spinor ϵ of $Spin(1,p) \times Spin(d_T)$:

$$\begin{aligned}
\delta\chi &= \Gamma^{MN}\epsilon F_{MN} \\
&= \frac{1}{2}\left(\Gamma^{MN}F_{MN} + \Gamma^{MN}\Gamma^{6789}\tilde{F}_{MN}\right)\epsilon \\
&= \Gamma^{MN}F_{MN}\left(1 + \Gamma^{6789}\right)\epsilon
\end{aligned}$$

So if

$$\epsilon = -\Gamma^{6789}\epsilon \tag{13.1}$$

then for the state $|\Psi(A)\rangle_{YM(\mathcal{B}_{p+4})}$

$$\delta_\epsilon\chi = \langle\Psi(A)|\{\epsilon\cdot Q, \chi\}|\Psi(A)\rangle = 0 \tag{13.2}$$

That is: spinors satisfying (13.1) define supercharges $\epsilon \cdot Q$ in the QFT $YM(\mathcal{B}_{p+4})$ which annihilate the instantonic-soliton. Therefore, these solitons preserve at least (in general, exactly) half the supersymmetry.

Moreover, there is a moduli space of these BPS states so

$$\mathcal{M}_{BPS} = \mathcal{M}^+ = \{A : F^+(A) = 0\}/Map[\mathbb{R}^4_{6789} \to U(w)]$$

14. How gauge fields change the RR charge of a Dp-brane

We argued that D-branes carry charge. This means that the SUGRA action in the presence of a brane must have something like the form:

$$\frac{1}{2}\int_{\mathbb{R}^{1,9}} dC^{(p+1)} \wedge *dC^{(p+1)} + \mu_p \int_{\mathcal{W}_{p+1}} C^{(p+1)}$$

and the presence of the brane modifies the equations to:

$$\begin{aligned}
dG^{(p+2)} &= 0 \\
d * G^{(p+2)} &= (-1)^{p+1}\mu_p \eta(\mathcal{W}_{p+1} \hookrightarrow \mathcal{S})
\end{aligned} \tag{14.1}$$

μ_p is the charge of the Dp-brane. Evidently, it should be proportional to the rank N of the CP bundle $E \to \mathcal{W}_{p+1}$.

14.1. Modification in the presence of gauge fields on the brane

The presence of gauge fields on the brane modifies t(14.1) in an important way.

We want to dodge several tricky issues so we just state what we believe is the correct result.

Let:

$G = \sum_p dC^{(p+1)}$ be the total RR fieldstrength

$\iota : \mathcal{W}_{p+1} \hookrightarrow \mathcal{S}$ the embedding of the brane worldvolume into spacetime.

$E \to \mathcal{W}_{p+1}$ the CP bundle, with typical fiber $V \cong \mathbb{C}^N$.

$\mathcal{F} \equiv F + \iota^*(B)1_N$ the modified fieldstrength.

$\mathrm{ch}\mathcal{F} \equiv Tr_V \exp(\frac{i\mathcal{F}}{2\pi})$

Then we claim that the equation of motion and Bianchi identity are unified into the single equation:

$$dG = \iota_*(\mathrm{ch}\mathcal{F}) \tag{14.2}$$

where ι_* is the push-forward defined using Poincare duality.

Heuristically, we write this in terms of δ-functions over the coordinates x^i transverse to $\mathcal{W}$:

$$dG = \prod_{i=1}^{d_T} \delta(x^i)dx^i \cdot Tr_V \exp(\frac{i\mathcal{F}}{2\pi})$$

The derivation of (14.2) is based on the "inflow argument" associated with a configuration of orthogonally intersecting D-branes:

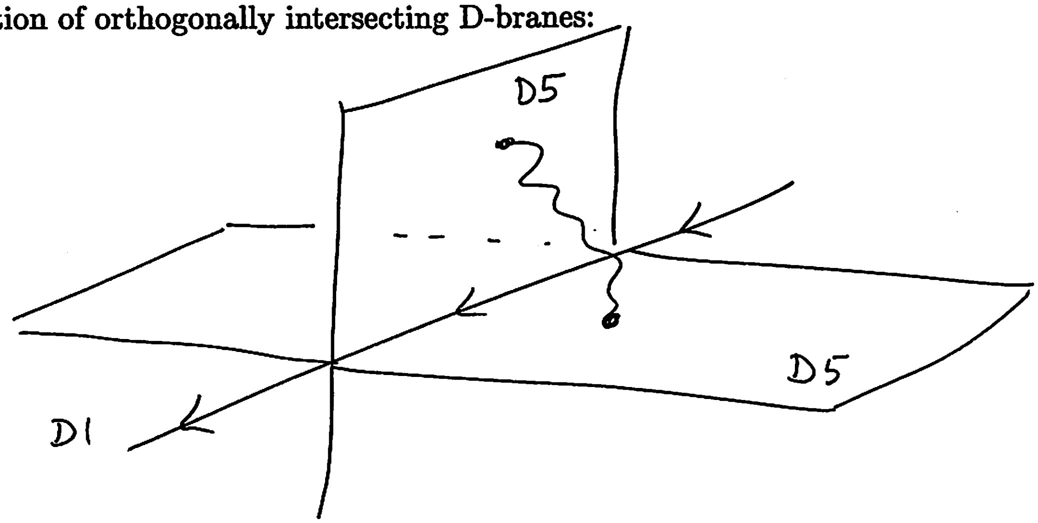


Fig. 27: Intersecting D-branes.

Remarks. There are several tricky points in this formula, among them:

1. The push-forward ι_* is usually only defined on compactly supported cohomology, we need it on the space of currents.
2. Formally the interaction Lagrangian must be of the form:

$$\int_{\mathcal{W}_{p+1}} \iota^* C \wedge Tr_{\mathcal{V}} e^{\mathcal{F}} \tag{14.3}$$

However, this involves both electric and magnetic potentials in the same Lagrangian, so its status as a Lagrangian in local QFT is confusing. So we stated it in terms of the equations of motion.

3. There are also gravitational contributions to (14.3) we are ignoring here.
4. When one combines (14.3) with the DBI action something nice happens. We consider G, B, ϕ, C as fixed background fields and considers the Lagrangian: [8]

$$\ell_s^{-p-1} \int_{\mathcal{W}} e^{-\phi} \sqrt{\det_{0 \le \mu,\nu \le p} \left[G_{\mu\nu} + \ell_s^2(\mathcal{F}_{\mu\nu}) \right]} + \int_{\mathcal{W}_{p+1}} \iota^* C \wedge Tr_{\mathcal{V}} e^{\mathcal{F}} \tag{14.4}$$

as a Lagrangian for a fundamental p-brane. When the background equations G, B, ϕ, C satisfy the supergravity equations of motion then the exact action (14.4) has an important *local supersymmetry* called κ-supersymmetry. It is thought to be essential to making a self-consistent theory of p-branes.

14.2. Brane charges and instantonic solitons

Let us now return to the anomalous couplings on a Dp-brane.

As a special case of (14.3) we get the interaction:

$$\int_{\mathcal{W}_{p+5}} C^{(p+4)} ch_0(\mathcal{F}) + C^{(p)} ch_2(\mathcal{F})$$

Let us choose a coordinate plane $\mathcal{B}_p \subset \mathcal{B}_{p+4}$ such that:

$$\mathcal{B}_{p+4} = \mathcal{B}_p \times \mathbb{R}^4_{6789}$$

Let us furthermore consider a gauge field configuration in the theory $YM(\mathcal{B}'_{p+4})$ such that

$$\int_{\mathbb{R}^{6789}} \mathrm{Tr} F \wedge F = 8\pi^2 v$$

[8] Actually, its supersymmetric version. Fermions terms are suppressed here for simplicity.

A $(p+4)$-brane with a $U(w)$ brane gauge field with instanton number v i.e., a Chan-Paton bundle with characteristic classes: $(ch_0 = w, ch_2 = v)$ has the same RR charge as a composite of w $(p+4)$-branes and v p-branes.

This suggests that just the way Dp-branes are the microscopic description of SUGRA solitons, "branes within branes" give the microscopic description of the instantonic solitons on Dp-branes.

In the next sections we will verify that this is correct.

15. The $(p, p+4)$ system

Let us now consider two parallel Dp branes.

$$(\mathcal{B}'_{p+4}, \mathbb{C}^w) \parallel (\mathcal{B}_p, \mathbb{C}^v)$$

Note there are two cases:

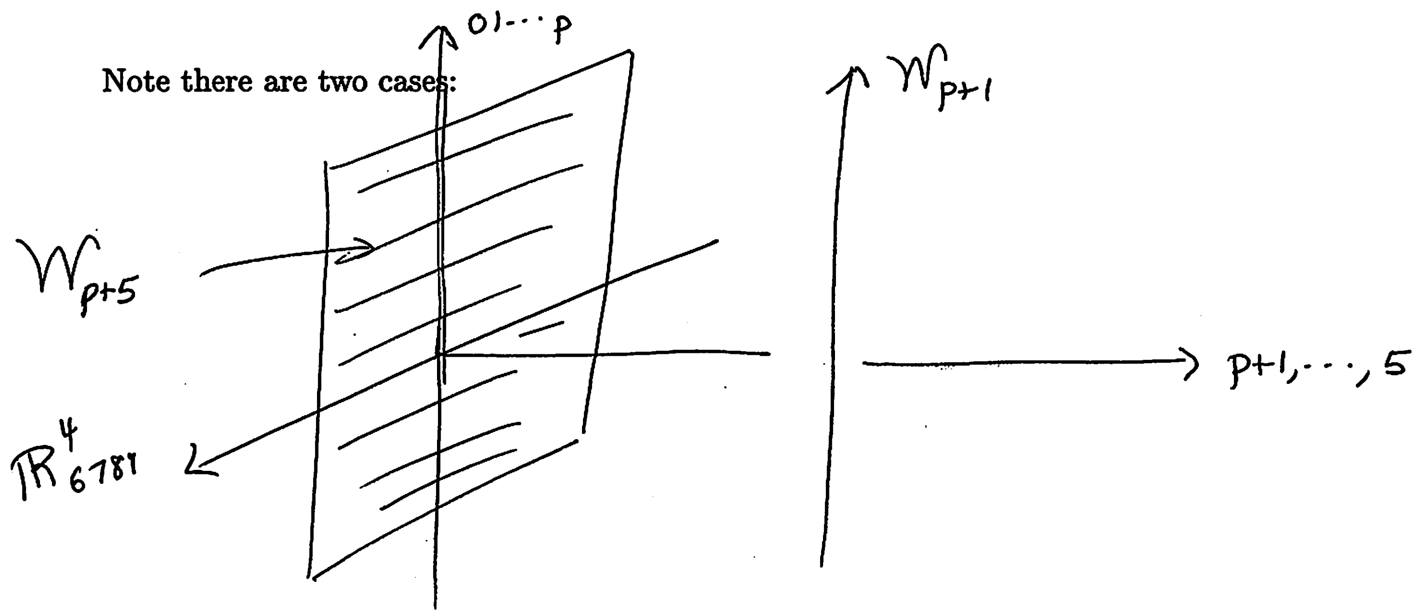


Fig. 28:

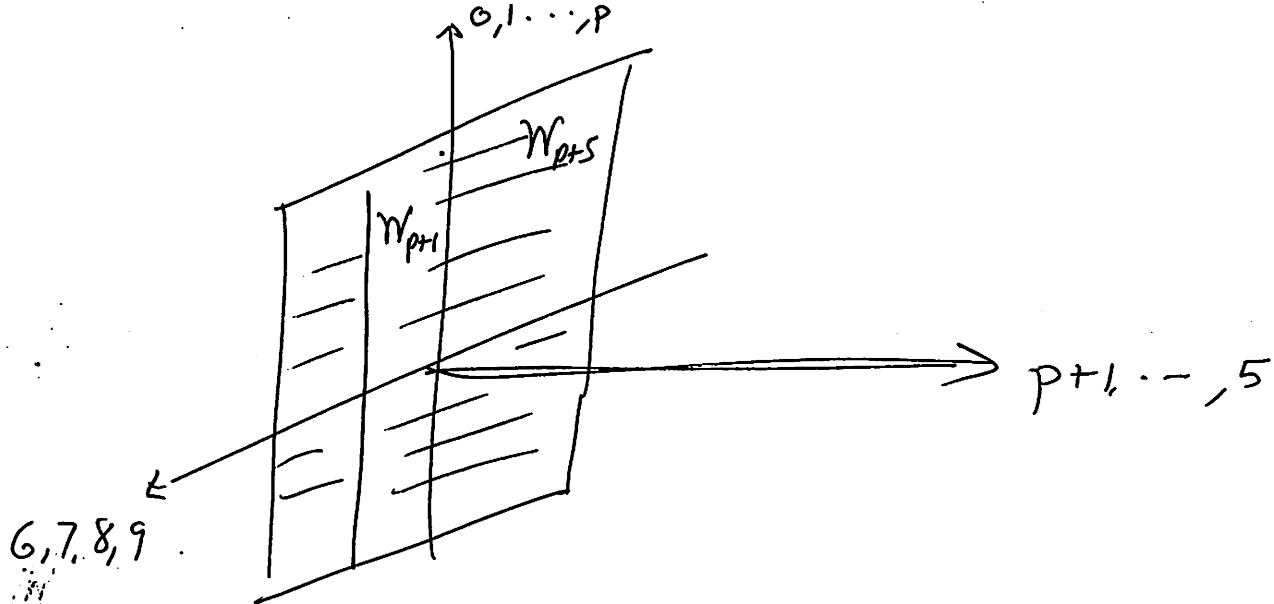


Fig. 29: $\mathcal{B}_p$ can lie within or outside $\mathcal{B}' = \mathcal{B}_{p+4}$.

The global rotational symmetries of this configuration are:

$$
\begin{aligned}
Spin(1,9) \supset & Spin(1,p) \times Spin(9-p) \\
\supset & Spin(1,p) \times Spin(5-p) \times Spin(4)_{6789} \\
p=5: \quad & Spin(1,5) \times Spin(4)_{6789} \\
p=4: \quad & Spin(1,4) \times Spin(1) \times Spin(4)_{6789} \\
p=3: \quad & Spin(1,3) \times Spin(2) \times Spin(4)_{6789} \\
p=2: \quad & Spin(1,2) \times Spin(3) \times Spin(4)_{6789} \\
p=1: \quad & Spin(1,1) \times Spin(4)_{2345} \times Spin(4)_{6789} \\
p=0: \quad & Spin(1,0) \times Spin(5) \times Spin(4)_{6789}
\end{aligned}
\tag{15.1}
$$

This configuration preserves 1/4 of the supersymmetries, namely, by the rule (11.2) we must have

$$
\begin{aligned}
\tilde{\epsilon} &= \Gamma^{01\cdots p}\epsilon \\
&= \Gamma^{01\cdots p}\Gamma^{6789}\epsilon
\end{aligned}
$$

The first equation eliminates $\tilde{\epsilon}$, leaving $\epsilon \in \mathbf{16}_+$ free. The second equation isolates a $\dim_R = 8$ subspace $S^+ \subset \mathbf{16}_+$ defined by the condition $\epsilon = \Gamma^{6789}\epsilon$.

Thus, the unbroken supersymmetry algebra is the minimal superalgebra $\mathcal{SP}(1,5)$ of six dimensions. We denote its dimensional reductions by $\mathcal{SP}_8(1,p)$.

The supersymmetry operators in $\mathcal{SP}_8(1,p)$ transform under the Lorentz $\times$ $\mathcal{R}$-symmetries (15.1) as:

$$
\begin{aligned}
p=5: \quad & [(\mathbf{4};\mathbf{2}_+) \oplus (\bar{\mathbf{4}};\mathbf{2}_+)]_{\mathbb{R}} \\
p=3: \quad & \cong (\mathbf{2},\mathbf{1};\mathbf{1}_{+\frac{1}{2}};\mathbf{2}_+) + (\mathbf{1},\mathbf{2};\mathbf{1}_{-\frac{1}{2}};\mathbf{2}_+)
\end{aligned}
\tag{15.2}
$$

15.1. *Examples of recent use*

All examples but 1,5, require further machinery for exposition, but we note that this system has played an important role in the past few years in many different examples:

1. $p=5$: Physical derivation of ADHM moduli space. (described below).
2. $p=4$: $D4 \parallel D8$: Extremal transitions in CY 3-folds.
3. $p=3$: $D3 \parallel D7$: F-theory.
4. $p=2$: $D2 \parallel D6$: 3D mirror symmetry.
5. $p=1$: $D1 \parallel D5$: Models of black holes; "little strings"
6. $p=0$: Matrix formulation of $(2,0)$ superconformal theories.

Each case has its own personality.

16. Summary of theories with 8 supercharges

16.1. *Superalgebra*

These come from 6D supersymmetry $\mathcal{SP}(1,5)$ above.

The supercharges are in the S^- of $Spin(1,5)$. [9] Since $\Lambda^2 S^-$ contains the vector we must introduce the symplectic Majorana condition: That is: the superalgebra contains the bosonic group:

$$Spin(1,5)\times USp(2)_R$$

with Q transforming in $S^-\otimes \mathbf{2}$, subject to a reality condition:

$$\begin{aligned}\{Q_{\dot\alpha r},Q_{\dot\beta s}\} &= +(C\gamma^M)_{\dot\alpha\dot\beta}P_M J_{rs}\\ Q^\dagger &= -B\otimes J\cdot Q\end{aligned}\tag{16.1}$$

16.2. *Massless reps of* $\mathcal{SP}(1,5)$

Massless reps are classified by the reps of the massless little group $Spin(4)_{1234}\times SU(2)$:

$$\begin{aligned}\pi_h &= (1,1;2)\oplus(2,1;1)\\ \pi_v &= [(2,2;1)\oplus(2,1;2)]_{\mathbb{R}}\\ \pi_t &= (3,1;1)_{\mathbb{R}}\oplus(1,1;1)\oplus(2,1;1)\end{aligned}\tag{16.2}$$

Bosonic fields:

---

9 S^- of $Spin(6)$ is $\cong$ $\mathbf{4}$ of $SU(4)$

HM: A quaternionic field Φ - 4 real scalars.

VM: A vector field

TM: not discussed

Field content for dimensions $p < 5$ follows from straightforward dimensional reduction.

$$
\begin{aligned}
& p=5 \to p=4 \to p=3 \to p=2 \\
& HM \to HM \to HM \to HM \\
& VM \to VM \to VM \to VM \cong (HM)^* \to HM \\
& TM \to VM \to etc
\end{aligned}
$$

16.3. 6D Low energy Lagrangian

For theories with 8 supercharges with a VM and linear HM's the low-energy effective Lagrangian in 6D is completely determined by the data:

- G - a compact Lie group with invariant bilinear form on $\underline{g}$.
- A quaternionic representation: $V \oplus V^*$ of G.
- FI parameters $\vec{\zeta} \in Center(\underline{g}) \otimes \mathbb{R}^3$.

$$
\frac{1}{g_{SYM}^2} \int_{\mathbb{R}^{1,5}} d^6\xi \left[\mathrm{tr}\ F_{\mu\nu}F^{\mu\nu} + \mathrm{tr}\ \bar{\chi} D \chi + (D\Phi, D\Phi)_V + (\bar{\psi}, D\psi)_V + \sum_{\Lambda=1}^{rankG} (\vec{\mu}^\Lambda - \vec{\zeta}^\Lambda)^2 \right] \quad (16.3)
$$

Here $\vec{\mu}^\Lambda$ is the HK moment map:

$$
\vec{\mu} \in \underline{g} \otimes \mathbb{R}^3
$$

for the G-action on the HM's.

Remark. $\vec{\mu}$ is often referred to as a "D-term." The D-terms of $d = 4, \mathcal{N} = 1$ susy are perhaps more familiar. The $USp(2)$ automorphism shows that any unit spinor ϵ^r defines an $\mathcal{N} = 1$ algebra. Thus, 3 independent D terms.

16.3.1. Reminder on HK Moment map

Choose coordinates: $\{z^\alpha\}_{\alpha=1,\ell}$ for V and dual coordinates w_α for V^* then we define quaternionic coordinates,

$$
\mathbf{X}^\alpha = \begin{pmatrix} z^\alpha & \bar{w}^\alpha \\ -w_\alpha & \bar{z}_\alpha \end{pmatrix} \quad (16.4)
$$

so that the complex structures $\mathbf{I}, \mathbf{J}, \mathbf{K}$ correspond to right multiplication by $i\sigma_3, i\sigma_2, i\sigma_1$, respectively. Moreover, G acts via

$$\delta_\Lambda(z^\alpha; w_\alpha) = \{(T_\Lambda)^\alpha{}_\beta z^\beta; -w_\alpha(T_\Lambda)^\alpha{}_\beta\} \tag{16.5}$$

where $1 \leq \Lambda \leq \dim G$ is an index labelling a basis for $\underline{g}$, and $[(T_\Lambda)^\alpha_\beta]^* = -(T_\Lambda)^\beta{}_\alpha$. This action may be written as:

$$\delta_\Lambda \mathbf{X}^\alpha = (\tau_\Lambda)^\alpha{}_\beta \mathbf{X}^\beta \tag{16.6}$$

where we replace T by $\mathrm{Re}T \cdot \mathbf{1} + \mathrm{Im}T \cdot \mathbf{I}$:

$$(\tau_\Lambda)^\alpha{}_\beta = \begin{pmatrix} (T_\Lambda)^\alpha{}_\beta & 0 \\ 0 & [(T_\Lambda)^\alpha_\beta]^* \end{pmatrix}. \tag{16.7}$$

For the vector space $V \oplus V^*$ we have Kahler and holomorphic symplectic forms

$$\begin{aligned} \omega^R &= \frac{i}{2}\sum[dz^\alpha d\bar{z}_\alpha + dw_\alpha d\bar{w}^\alpha] \\ \omega^C &= \sum dz^\alpha \wedge dw_\alpha \end{aligned} \tag{16.8}$$

these forms comprise a triplet $\vec{\omega}$. The G-action is symplectic with respect to each of these forms and hence we obtain a triplet of Noether charges defining the hyperkähler moment map:

$$\vec{\mu}_\Lambda = \frac{1}{2}\mathrm{tr}\ \vec{\sigma}\mathbf{X}^\dagger_\alpha (\tau_\Lambda)^\alpha_\beta \mathbf{X}^\beta \quad . \tag{16.9}$$

Explicitly:

$$\begin{aligned} \mu^R_\Lambda &= \frac{1}{2}\left[\bar{z}_\alpha (T_\Lambda)^\alpha_\beta z^\beta - w_\alpha (T_\Lambda)^\alpha{}_\beta \bar{w}^\beta\right] \in \sqrt{-1}\mathbb{R} \\ \mu^C_\Lambda &= w_\alpha (T_\Lambda)^\alpha_\beta z^\beta \end{aligned} \tag{16.10}$$

17. Low energy spectrum on the two branes in the $(p, p+4)$ system

We now return to our system:

$$(\mathcal{B}'_{p+4}, \mathbb{C}^w) \parallel (\mathcal{B}_p, \mathbb{C}^v)$$

Global bosonic symmetries:

$$Spin(1,p) \times Spin(5-p) \times Spin(4)_{6789} \times U(w) \times U(v)$$

17.1. Massless modes on the D(p+4)-brane

This is just the reduced SYM multiplet for $U(w)$.

$$(A_M)^m{}_{m'} \qquad M = 0, \ldots, 9; m, m' = 1, \ldots, w$$

$$(\chi_{\alpha A})^m{}_{m'}$$

17.2. Massless modes on the Dp-brane from $\mathcal{H}_{(\mathcal{B}_p,\mathcal{B}_p)}$

Result: As a representation of the algebra $\mathcal{SP}(1,5)$ we get 1 VM and 1 HM in the adjoint $u(v)$.

Proof:

This is just the reduced SYM multiplet for $U(v)$:

$$\sum_{\mu=0}^{p} B_\mu dx^\mu \in \mathcal{A}(\mathcal{W}_{p+1}; u(v))$$

$$B_{6,7,8,9} \in \Omega^0(\mathcal{W}_{p+1}; u(v))$$

$$\chi \in \Gamma[S^+ \otimes u(v)]$$

with CP indices:

$$(B_M)^j{}_{j'}, \chi^j{}_{j'} \qquad j, j' = 1, \ldots, v$$

However, because of interactions with the external fields on $\mathcal{B}_{p+4}$ we split the $\mathcal{SP}_{16}(1,p)$ VM to the $\mathcal{SP}_8(1,p)$ representation VM $\oplus$ HM:

$$p = 5: \qquad B_M \to (B_{M=0,1,\ldots,5})_{VM} \oplus (B_{M=6789})_{HM}$$

$$p = 3: \qquad \to (B_{\mu=0,1,\ldots,3}, \phi_{a=4,5})_{VM} \oplus (B_{M=6789})_{HM}$$

17.2.1. The mixed sector $\mathcal{H}_{(\mathcal{B},\mathcal{B}')}$

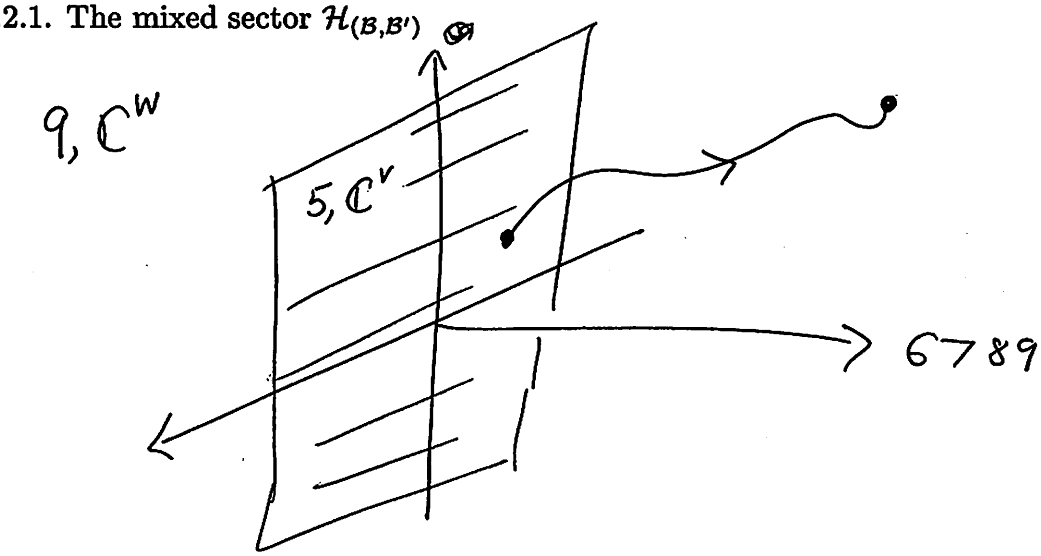


Fig. 30: (5,9) fields

This gives the most interesting fields.

Result: As a representation of $\mathcal{SP}(1,5)$ this sector gives 1 HM in the $(v,\bar{w})$ of $U(v) \times U(w)$.

Proof:

CFT boundary conditions:

	0,1,..., p	p+1,..., 5	6,7,8,9
$\sigma = 0$	N	D	D
$\sigma = \pi$	N	D	N

DN sector boundary conditions in the $(\mathcal{W}_{p+1}, \mathcal{W}_{p+5})$ system.

Referring to the above oscillator calculations we see that the groundstate energy $E = 0$ in the NS sector.

Moreover, in the NS sector, which gives *spacetime bosons*, we have integer index fermions $\psi_n^{6,7,8,9}$ and hence, the NS vacuum is a representation of the Clifford algebra:

$$\{\psi_0^\mu, \psi_0^\nu\} = -\delta^{\mu\nu} \qquad \mu,\nu = 6,7,8,9$$

The massless fields are described on the worldsheet and spacetime by:

W: After GSO projection the Clifford vacuum in the NS sector transforms in the $\mathbf{2}_+$ of the global symmetry $Spin(4)_{6789}$. The groundstates are $|A\rangle \otimes \lambda^m{}_i$

S: These are the bosonic fields in a *hypermultiplet*:

$$\tilde{h} \in \Gamma[(1;1;2,1) \otimes V_{\mathcal{B}'} \otimes V_{\mathcal{B}}^*]$$

$$\tilde{h}^{Am}{}_i$$

Since the momentum can only be nonzero in the directions $0,1,\ldots,p$ these complex fields are confined to the brane $\mathcal{B}_p$ (and *not* $\mathcal{B}'$).

Remark. Reality conditions. The sectors $\mathcal{A}_{(\mathcal{B},\mathcal{B}')}$ and $\mathcal{A}_{(\mathcal{B}',\mathcal{B})}$ correspond to the two orientations of strings. These statespaces can be mapped into each other by a hermitian conjugation. In the spacetime field theory $YM(\mathcal{B}_p)$ the fields are related by a reality condition:

$$\epsilon^{AB}\left(h^{Bi}{}_m(x)\right)^* = \tilde{h}^{Am}{}_i(x) \quad . \tag{17.1}$$

It is useful to define the notation:

$$\begin{aligned} J &= \tilde{h}^1 = (h^2)^\dagger \in \mathrm{Hom}(V,W) \\ I &= (\tilde{h}^2)^\dagger = -h^1 \in \mathrm{Hom}(W,V) \end{aligned} \tag{17.2}$$

for the VEV's of these fields

18. Physical derivation of the ADHM description of instanton moduli space

In the previous section we derived the low energy field content on the various branes. We are working with *two* gauge theories: one on the p-brane and one on the $(p+4)$-brane.

The instantonic soliton of the $YM(\mathcal{B}'_{p+4})$ theory has a microscopic description in terms of the $YM(\mathcal{B}_p)$ theory.

Let us begin with the case $p = 5$. [10]

18.1. *The classical action*

In six dimensions g^2_{YM} has dimensions of $[LENGTH]^2$. At distances large compared to this length (16.3) is a good approximation.

The classical $6D$ action is completely determined by the field content, and this is exactly what we solved for in the previous section. [11]

Therefore, all we need to do is work out the precise content of the D-terms.

In the situation under consideration we can write the D terms as follows. We have 4 scalar fields B_ℓ valued in $u(v)$. The moment map transforms in the $(1,3)$ of $Spin(4)_{6789}$, identified with $S^2(S^+)$. Thus we can write:

$$(\mu_{AB})^i{}_j = [B_\ell, B_{\ell'}]^i{}_j \sigma^{\ell\ell'}_{AB} + (\tilde{h}_A)^i{}_m (h_B)^m_j + (\tilde{h}_B)^i{}_m (h_A)^m_j \tag{18.1}$$

Thus, the moduli space of supersymmetric vacua in the theory $YM(\mathcal{B}_p)$ on the p-brane. is just the finite dimensional HK quotient:

$$\begin{aligned} \mathcal{M}^{\text{brane}} &= \{(B,h,\tilde{h}) : [B_\ell, B_{\ell'}]^i{}_j \sigma^{\ell\ell'}_{AB} + (\tilde{h}_A)^i{}_m (h_B)^m_j + (\tilde{h}_B)^i{}_m (h_A)^m_j = 0\}/G \\ G &= U(v) \end{aligned} \tag{18.2}$$

---

[10] This is slightly unphysical because of quantum anomalies in the bulk theory. They are not relevant to this discussion. In a type I theory we cancel the anomalies with $U(w) \rightarrow Spin(32)/\mathbb{Z}_2$.

[11] $\vec{\zeta} = 0$ in our situation, and g^2_{YM} has been determined.

18.2. Comparing the formulations

Now let us put together what we have learned:

In the theory $YM(\mathcal{B}'_{p+4})$ we can consider the moduli of supersymmetric vacua associated with instantonic-solitons made from $U(w)$ instantons of $\mathrm{ch}_2 = v$. The macroscopic description of this moduli space is the infinite-dimensional HK quotient:

$$\begin{aligned} \mathcal{M}^{\text{bulk}} &= \{A_\mu(x) \in \mathcal{A}(\mathcal{W}_{p+5}; u(w)) : F_+(A) = 0\}/\mathcal{G} \\ \mathcal{G} &\equiv Map[\mathbb{R}^4_{6789} \to U(w)\} \end{aligned} \tag{18.3}$$

On the other hand, there is a microscopic description of the same moduli space: We can take w $(p+4)$-branes with v p-branes (embedded, in the Higgs branch) and consider the moduli space of supersymmetric vacua in the theory $YM(\mathcal{B}_p)$ on the p-brane. This leads to the finite dimensional HK quotient:

$$\begin{aligned} \mathcal{M}^{\text{brane}} &= \{(B, h, \tilde{h}) : B_\ell, B_{\ell'}]^i{}_j \sigma^{\ell\ell'}_{AB} + (\tilde{h}_A)^i{}_m (h_B)^m_j + (\tilde{h}_B)^i{}_m (h_A)^m_j = 0\}/G \\ G &= U(v) \end{aligned} \tag{18.4}$$

While the moduli spaces (18.3) and (18.4) appear very different they are merely two different descriptions of the same physical object: the space of low energy excitations of a Dbrane. Hence one suspects that they are in fact the same:

$$\mathcal{M}^{\text{brane}} = \mathcal{M}^{\text{bulk}} \tag{18.5}$$

This is indeed a mathematical fact: it is the main result of ADHM.

Remark. Global vs. gauge symmetries:

In the theory$YM(\mathcal{B}_p)$ $U(w)$ is global, $U(v)$ is gauge.

In the theory $YM(\mathcal{B}_{p+4})$: $U(w)$ is gauge. $U(v)$ is strongly broken by the instanton size - i.e., the vevs of the HM's $\tilde{h}$. It is not a symmetry of the low energy theory $YM(\mathcal{B}_{p+4})$. That is, " $U(v)$ is not visible because it is "hidden at the core of the soliton."

Remark For many purposes it can be useful to break the $Spin(4)_{6789}$ symmetry by choosing a complex structure for the transverse space $\mathbb{R}^4_{6789}$ we can define two complex scalar fields $B, \tilde{B}$ valued in $u(v)$:

$$\begin{aligned} B &= B_6 + iB_7 \\ \tilde{B} &= B_8 + iB_9 \end{aligned}$$

Then we also break the symmetry of the D-terms and get real and complex moment maps:

$$\begin{aligned} \mu^C &= [B, \tilde{B}] + IJ \\ \mu^R &= [B, B^\dagger] + [\tilde{B}, \tilde{B}^\dagger] + II^\dagger - J^\dagger J \end{aligned} \tag{18.6}$$

for the $U(v)$ gauge group

19. Measuring the instanton field

The above discussion shows that Dbranes can reproduce the ADHM description of the *moduli space* of D-branes. One can also recover the ADHM construction of the gauge field itself.

19.1. (0, 4) *sigma models*

(0, 4) susy allows leftmoving fermiosn λ_+ (and leftmoving bosons) which are invariant under supersymmetry, $\delta_\epsilon \lambda_+ = 0$.

Theorem[Howe-Papadopoulos, Witten]. A (0, 4) sigma model with leftmoving fermions λ^i can couple to a spacetime gauge field:

$$\int d^2\sigma \lambda_+ (\partial_+ + A_\mu \partial_+ X^\mu) \lambda_+ \tag{19.1}$$

only when A_μ is an instanton.

19.2. The (1, 5, 9) *system*

To do this properly we really should work in the Type I string.

Still-forgetting about certain quantum anomalies in the bulk theory we can understand the ADHM construction of the $U(w)$ gauge fields of instanton number v.

19.2.1.Constructing a (4, 4) string: The $(p, p+8) = (1, 9)$ system

$SO(1,9) \supset SO(1,1) \times SO(8)$

$16 = (+, 8_+) \oplus (-, 8_-)$

DD: 8 left and right-moving $(X^\mu(x^0, x^1), S^a(x^0, x^1))$ in the *internal* $8_v \oplus 8_+$.

Note: The fermion zeromodes are in the Green-Schwarz formalism in static gauge.

DN - sector

$(DN, NS) : E = 8(+\frac{1}{16}) > 0$ no massless modes

(DN, R): Only $\psi^{0,1}$ have zeromodes. Quantize Clifford: 2 states. GSO: one real state. (To see this use the $U(1) \to \mathbb{Z}_2$ worldsheet gauge symmetry.)

19.3. Three parallel branes: The 159 system

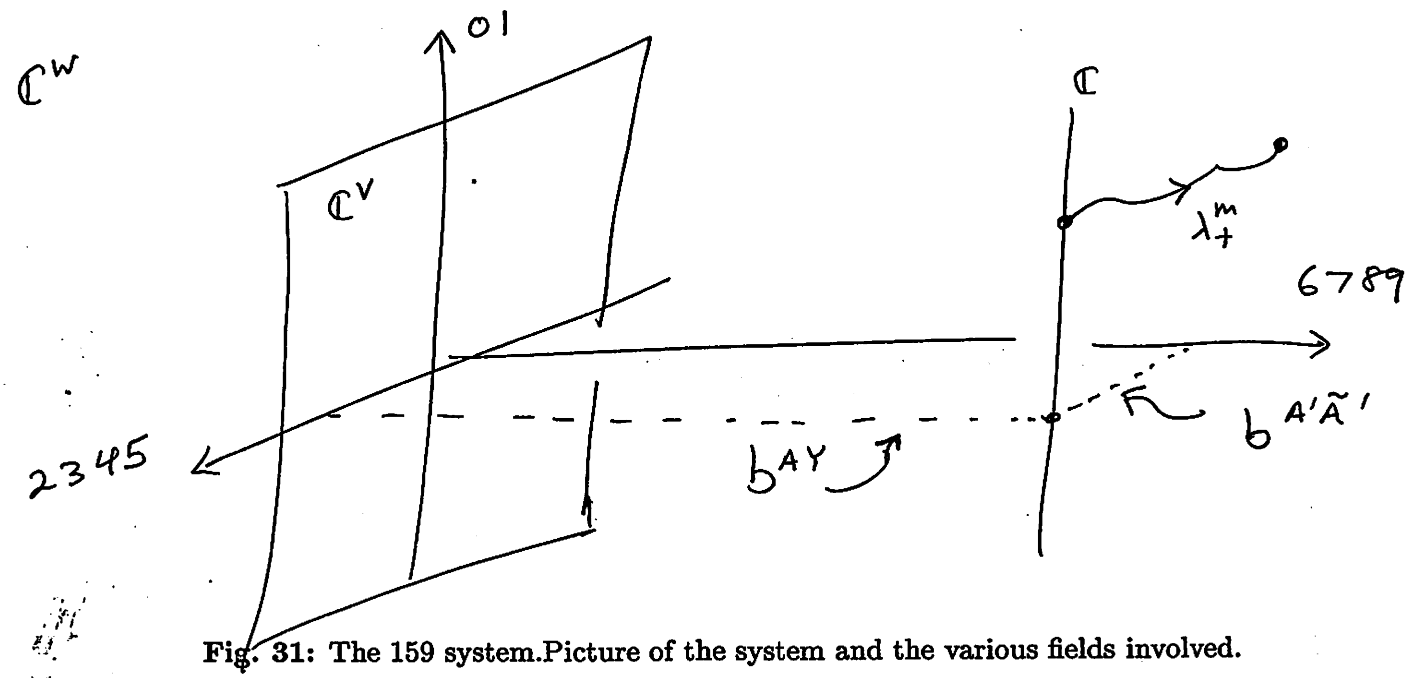


Fig. 31: The 159 system.Picture of the system and the various fields involved.

Combine the systems:
19: string + 59: instanton
new element 15, DN sectors
1 D1 brane along $X^{0,1}$
v D5 branes X^{012345} plane
w D9 branes
Global symmetries:

$$Spin(1,1)_{01} \times Spin(4)_{2345} \times Spin(4)_{6789}$$

Denote spinor indices: $(\pm; 2^{A'}, 2^{\tilde{A}'}; 2^{A}, 2^{Y})$

$$16 = (+, 2^A, 2^{A'}) \oplus (+, 2^Y, 2^{\tilde{A}'}) \oplus (-, 2^A, 2^{\tilde{A}'}) \oplus (-, 2^Y, 2^{A'})$$

Unbroken susy: $\epsilon = \Gamma^{01}\epsilon = \Gamma^{2345}\epsilon = \Gamma^{6789}\epsilon$

Preserve 1/8 of the susy's: The unbroken supersymmetry is of type $\epsilon^{+AA'}$ giving $(0,4)$ susy σ-model on $\mathcal{B}_1$.

(11) Sector:
$(b^{A'\tilde{A}'}, \psi_-^{A\tilde{A}'})$: transverse positions of $\mathcal{B}_1$ in the space $\mathbb{R}^4_{2345}$ parallel to $\mathcal{B}_5$ plane
$(b^{AY}, \psi_-^{YA'})$: transverse positions of $\mathcal{B}_1$ in the space $\mathbb{R}^4_{6789}$ transverse to $\mathcal{B}_5$ plane

(15) Sector:
$(\phi^{A'm}, \chi_-^{Am})$, χ_+^{Ym}.

(19) Sector:

We also get fields on the D1 probe from λ_+^m, $m = 1, \ldots, w$. Ω projection shows they are real.

(55) and (59) fields: are external, entering as parameters in the $\mathcal{B}_1$ Lagrangian. Recall that the vev's $(B_{6789})^i{}_j$ are moduli of the gauge field. These transform in the rep $(2,2)$ of $Spin(4)_{6789}$ and hence are conveniently denoted as $(B_{AY})^i{}_j$

Projecting out the massive modes:

Computing the $(0,4)$ sigma model Lagrangian using the above technique one arrives at the fermion terms:

$$L^{fermi\ mass} = \int_{\mathcal{B}_1\times\mathbb{R}} d^2\sigma \begin{pmatrix}\chi_1 & \chi_2\end{pmatrix}_- \cdot \begin{pmatrix} B^{11} - b^{11} & B^{12} - b^{12} & h_M^{1m} \\ B^{21} - b^{21} & B^{22} - b^{22} & (h^{2\ tr})_m^M \end{pmatrix} \begin{pmatrix} \chi^1 \\ \chi^2 \\ \lambda \end{pmatrix}_+ \tag{19.2}$$

Now - for the $(0,4)$ superconformal field theory we are only interested in the massless fermion zeromodes. So we should find the zeromodes ν of the operator in (19.2). Projecting these out gives a coupling of type (19.1) with the gauge field A given by the ADHM construction.

19.4. Appendix: The ADHM construction of the instanton

In the standard ADHM construction, The bundle and instanton are constructed as follows: One introduces linear operators:

$$\sigma = \begin{pmatrix} B + z_1 \\ \tilde{B} - z_2 \\ J \end{pmatrix} : V \to V^{(+)} \oplus V^{(-)} \oplus W \tag{19.3}$$

$$\tau = \begin{pmatrix} -(\tilde{B} - z_2) & B + z_1 & I \end{pmatrix} : V^{(+)} \oplus V^{(-)} \oplus W \to V \tag{19.4}$$

where $V^{(\pm)} \cong V$. In terms of these operators the ADHM equations read

$$\begin{aligned} \tau\sigma &= 0 \\ \sigma^\dagger\sigma - \tau\tau^\dagger &= 0 \end{aligned} \tag{19.5}$$

respectively. One may define the bundle holomorphically as $\ker\tau/im\sigma$, or, using the hermitian metric on V, W, as

$$\begin{aligned} E &\equiv \ker \mathcal{D}^\dagger \cong \mathrm{cok}\mathcal{D} \\ \mathcal{D} &\equiv \begin{pmatrix}\sigma & \tau^\dagger\end{pmatrix} : V \oplus V \to V^{(+)} \oplus V^{(-)} \oplus W \end{aligned} \tag{19.6}$$

If $P : V^{(+)} \oplus V^{(-)} \oplus W \to E$ is the orthogonal projection then the ASD connection on E is, conceptually, simply $\nabla = Pd$ where d is the exterior derivative acting on sections of the trivial bundle $V^{(+)} \oplus V^{(-)} \oplus W$.

20. The Higgs and Coulomb phases

When we consider the system:

$$(\mathcal{B}'_{p+4}, \mathbb{C}^w) \parallel (\mathcal{B}_p, \mathbb{C}^v)$$

with $p < 5$ an interesting and important new phenomenon arises.

It is best to start with the classical (16.3) and reduce to $p < 5$.

20.0.1.Hypermultiplet mass

An important term is $|D\tilde{h}|^2$. Written out with indices, in 6d this becomes:

$$(D_\mu \tilde{h})^{Am}{}_j = \partial_\mu \tilde{h}^{Am}_j + \tilde{h}^{Am}_i (B_\mu)^i{}_j$$

and hence, upon dimensional reduction to $p < 5$ we get a mass term:

$$\sum_{A=1,2} \sum_{m=1}^{w} \sum_{j=1}^{v} \sum_{a=p+1}^{5-p} |\tilde{h}^{Am}_i (B_a)^i{}_j|^2 = \mathrm{tr}\ _V(\phi\phi^\dagger J^\dagger J + \phi^\dagger \phi I I^\dagger)$$

Recall that the eigenvalues of ϕ just represent the separation between $\mathcal{B}$ and $\mathcal{B}'$. These mass terms are just the mass of the DN strings due to the stretching in the DN sector.

20.1. The supersymmetric vacua

Now consider the moduli space of vacua of the theory $YM(\mathcal{B}_p)$.

Already from the classical Lagrangian we see something significant. The moduli space $\mathcal{M}^{brane}$ of zero energy states in the Dp theory is obtained from:

$$\mathrm{tr}\ _V(\phi\phi^\dagger J^\dagger J + \phi^\dagger \phi I I^\dagger) = 0$$

$$\vec{D}^I = \vec{\zeta}^I \qquad \mathrm{mod} U(v)$$

Now there are *two* very different branches of the supersymmetric vacua.

Coulomb: $\phi^\dagger \phi \neq 0$. The branes are separated: $\Rightarrow I = J = 0$.

Higgs: $\phi^\dagger \phi \neq 0$. $\mathcal{B}_p \subset \mathcal{B}_{p+4}$

Remark: So far this is a classical picture. The next step uses extended supersymmetry. *The Higgs branch moduli space is not altered by quantum or string corrections.* Of course, the Coulomb branch is, this is the subject of SW theory. In the present context there are many confusing points, currently one of the most intensively studied issues.

20.2. The small instanton transition

The eigenvalues of $II^\dagger$, $J^\dagger J$ measure the sizes of the instantons.

For example when $v = 1$ the equations become:

$IJ = 0$ $II^\dagger = J^\dagger J$ mod$U(1)$.

The instanton scale size is thus identified with the HM vev $\tilde{h}h$. This gives a type II string theory answer to the question:

"What happens to a small instanton? "

The suggested picture is a "phase transition":

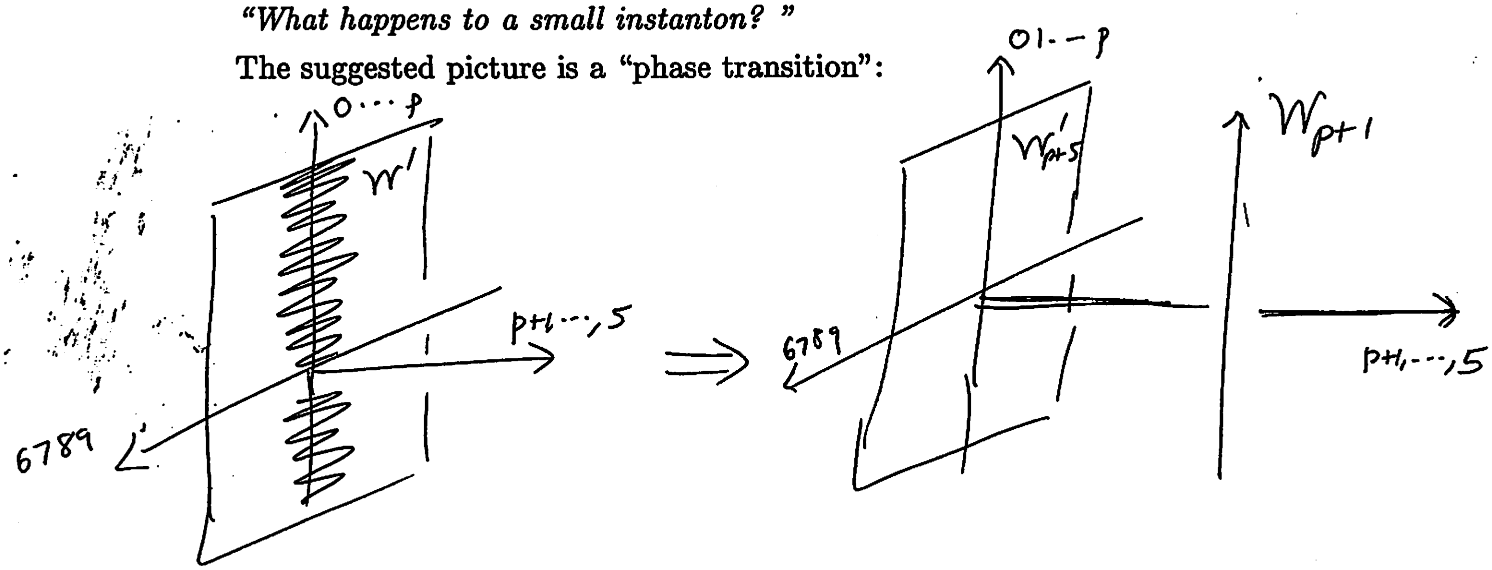


Fig. 32: A small instanton escapes from the brane.

20.3. 0-branes and SQM on the moduli space of instantons

Remark: It is worth noting that for $p = 0$ the Higgs branch consists of SQM with target space the HK quotient. Now, SQM gives a realization of Hodge theory. The *Spin*5 symmetry, which is obvious from the brane setup becomes an $USp(4) \cong Spin5$ symmetry of the cohomology of the moduli space. Similarly, the Lefshetz $SL(2)$ symmetry of Kahler manifolds can be realized as a global symmetry brane a brane picture.

21. Some other topics along these lines

21.1. D-branes and the Nahm transform

D-branes give some new ways of looking at instanton physics. For example -

1. The generalization to instantons on ALE spaces is straightforward. One can reproduce the results of Kronheimer-Nakajima.
2. Some aspects of the Nahm transform follow simply from T-duality.
3. Calorons, Nahm equations for monopoles, ...

21.2. *Matrix theory*

Many of the above techniques/ideas underly the Matrix theory proposal. c.f. Dijkgraaf's lectures.

21.3. *Probing metrics*

Probing ALE geometry - it works.

Will Dbranes give an explicit K3 metric?

21.4. *D-branes and black holes*

Dbranes provided interesting tractable models of quantum black holes.

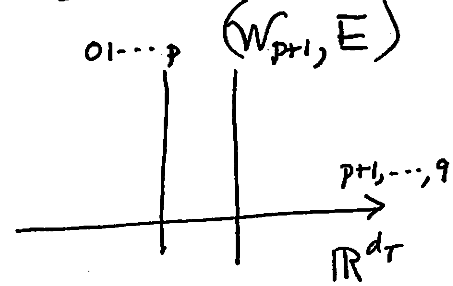
An interesting open puzzle: D-branes have trouble "seeing" the horizon of a black hole in current models.

21.5. *Theories on branes*

Further development of the Matrix theory proposal appears to rely on understanding theories of D-branes decoupled from the bulk SUGRA. There are interesting proposals to formulate the $(0,2)$ superconformal theory in $\mathbb{R}^{1,5}$ in terms of quantum mechanics on instanton moduli space.

Lecture 3

Last time we saw the following



1. Dbrane statespace $\mathcal{A}_{B,B}$:

$$\mathbb{C}^N \to E$$
$$\downarrow$$

data: $\begin{cases} \text{vector bundle} \\ \text{coord. subspace } W \end{cases}$

2. A low energy state of the Dbrane is specified by the data

$$A \in \mathcal{A}(W_{p+1}; u(N))$$

$$\phi_a \in \Omega^0(W_{p+1}; u(N))$$

fields in $YM(W_{p+1}, E)$ $|\Psi(A,\phi,\cdots)\rangle \in \mathcal{H}_{YM(W_{p+1})}$

3. Groundstates of the brane field theory:

$$V(\phi) = \sum_{a<b} \mathrm{Tr}([\phi_a,\phi_b])^2 \Rightarrow F(A) = 0, \quad \phi_a \sim \begin{pmatrix} \phi_a^1 & & \\ & \ddots & \\ & & \phi_a^N \end{pmatrix} \text{ mod Weyl gp.}$$

$$\therefore \quad \mathcal{M}_{\text{Vacua}} = S^N \mathbb{R}^{d_T}$$

$$= \{ \langle \phi \rangle \text{ in } YM(W_{p+1}) \}$$

4. Subtle point: the data

$$|\Psi(\mathcal{N}, E, A, \phi, \cdots)\rangle_{SFT(\mathbb{R}^{1,9})}$$

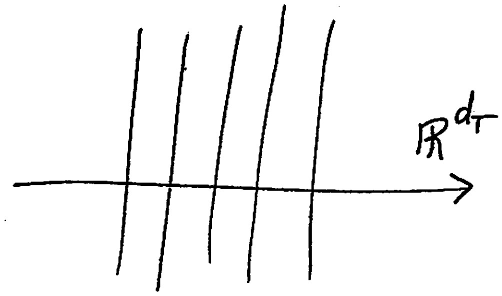
also specify a state in 10D sugra

When $F(A)=0$ $V(\phi)=0$, we will see it is a BPS state:

$$\begin{cases} G_{00} = -\left(1 - g_s\left(\frac{l_s}{r}\right)^{d_T-2} + \cdots\right) \\ \text{etc.} \end{cases}$$

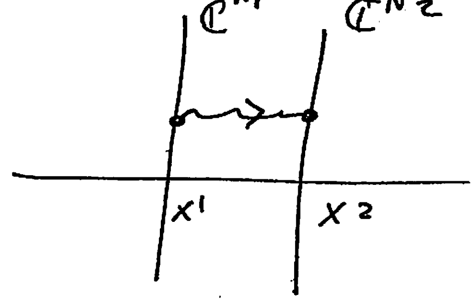
$$\mathcal{M}_{BPS} = S^N \mathbb{R}^{d_T}$$

$= \text{Positions of branes}$



Comparison of the two leads to a beautiful geometrical interpretation of the adjoint Higgs mechanism:

$$\langle \vec{\phi}_a \rangle = \begin{pmatrix} \vec{\phi}^1_a \mathbb{1}_{N_1} & 0 \\ 0 & \vec{\phi}^2_a \mathbb{1}_{N_2} \end{pmatrix}$$



brane　　　　　　bulk

$$\phi^1_a - \phi^2_a = \frac{(\Delta X)^a}{l_s^2}$$

5. Today:

- Add some susy to justify some statements
- Consider instantons $A \in \mathcal{A}(W_{p+5}; E)$
- These will likewise have two spacetime descriptions

$$\mathcal{M}_{BPS} = \{ A \mid F^{+}(A) = 0 \} / \mathrm{Map}(W_{p+5} \to U(W))$$

$$\mathcal{M}_{susy\,vac.} = \{ (\phi, h) \mid \vec{\mu}(\phi, h) = 0 \} / U(V)$$

these are two descriptions of the same physical space, giving a new picture of the ADHM theorem identifying the two.

- Discuss briefly applic's + future directions.

L3-3

The bosonic strings and their Dbranes discussed above are not really consistent. We need supersymmetry. From now on we work in 10 dimensional superstrings. $s = 9$.

6. Superconformal field theory with a boundary

Action:

$$S = \frac{1}{4\pi}\int_\Sigma \ell_s^{-2}\partial X^\mu \bar\partial X_\mu + \psi^\mu \bar\partial \psi_\mu + \tilde\psi^\mu \partial \tilde\psi_\mu$$

We start with $\Sigma =$ the strip.

Boundary conditions for bosons are chosen as D or N as before. we must relate ψ to $\tilde\psi$

For the ws fermions $\psi, \tilde\psi$, Since we have first order systems ~~the boundary reflects R-movers into L-movers~~. Two ~~possible~~ inequivalent boundary conditions on the fermions are:

$$R: \psi^\mu(\sigma^1 = 0) = \tilde\psi^\mu(\sigma^1 = 0)$$
$$\psi^\mu(\sigma^1 = \pi) = \tilde\psi^\mu(\sigma^1 = \pi)$$
$$NS: \psi^\mu(\sigma^1 = 0) = -\tilde\psi^\mu(\sigma^1 = 0)$$
$$\psi^\mu(\sigma^1 = \pi) = \tilde\psi^\mu(\sigma^1 = \pi)$$

Since modular tmns mix spin structures, A consistent superconformal field theory requires introducing both boundary conditions so that the CFT statespace has a $\mathbb{Z}_2$-grading:

$$\mathcal{A} = \mathcal{A}_{NS} \oplus \mathcal{A}_R$$

In addition to the choices of N, D at the two boundaries.

6.1. Oscillators and quantization

Quantization of the oscillator expansions associated with the above bc's gives a Heisenberg and a Clifford algebra:

$$[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{n+m,0}\eta^{\nu\mu}$$
$$\{\psi^\mu_r, \psi^\nu_{r'}\} = \delta_{r+r',0}\eta^{\mu\nu} \tag{6.1}$$

~~Modings:~~
~~(D,D) and (N,N):~~
~~NS: $(\alpha_n, \psi_{n+\frac{1}{2}})$~~

Here $r \in \mathbb{Z}$ or $r \in \mathbb{Z}+1/2$. The various modings depend on the b.c.'s

	NS	R
NN	$(\alpha_n, \psi_{n+1/2})$	(α_n, ψ_n)
DD	$(\alpha_n, \psi_{n+1/2})$	(α_n, ψ_n)
DN, ND	$(\alpha_{n+1/2}, \psi_n)$	$(\alpha_{n+1/2}, \psi_{n+1/2})$

Must represent:

Main new ingredient: Choose (N,N) for simplicity.

$N=0$ states [redacted] in $\mathcal{H}_R$ form rep$^{\underline{n}}$ of

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$$

Choose irrep: [redacted] $16_+ \oplus 16_-$ [redacted]

$16_\pm$ MW reps of Spin(1,9).

$$\mathcal{H}_{NS} = \int dp \ \mathcal{F}_p$$

$$\mathcal{H}_R = \int dp \ \mathcal{F}_p \otimes (16_+ \oplus 16_-)$$

$\mathcal{F}_p$ = Fock space — [redacted] Symmetric algebra on α_{-r}, ψ_{-r}

$\mathrm{Symm}[\alpha_{-r}, \psi_{-r}]$

Superstrings

$\mathcal{S} = \mathbb{R}^{1,9}$ as before we start with (N,N) on all (X^μ, ψ^μ)

We want an action of $SO_{16}(1,9)$ on $\mathcal{A}$ – clearly it is quite asymmetric.

But: $\exists$ ∞-dim'l chirality operator assoc. w/ Clifford

$$(-1)^F = (-1) \cdot \prod_{r>0} (-1)^{\psi_{-r}\psi_r} \qquad NS$$

$$= \psi_0^0 \cdots \psi_0^9 \prod_{r>0} (-1)^{\psi_{-r}\psi_r} \qquad R$$

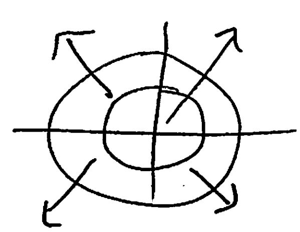
Thm 1: The chirally projected theories

$$\mathcal{A}^{\pm} \equiv \mathcal{A}_{NS}^{+} \oplus \mathcal{A}_R^{\pm}$$

are consistent CFT's.

Remark: $\mathcal{A}_R^{\pm}\big|_{N=0} \cong 16_{\mp}$ of $Spin(1,9)$

To explain the next point, suppose we work with just $X^\mu(z), \psi^\mu(z)$

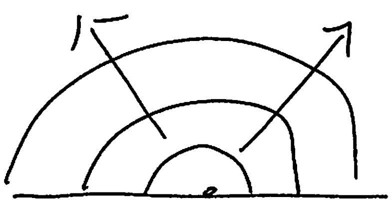


Thm 2 In $\mathcal{A}^{\pm}_{S^1}$ there are local Virasoro primaries

$$j_\alpha(z) \qquad |\alpha\rangle \in 16_{\pm}$$

such that $Q_\alpha = \oint j_\alpha(z)$ generates an action of $SP_{16}(1,9)$ on $\mathcal{A}^{\pm}$.

That's not actually our problem – we have



$\psi, \tilde{\psi}$

You can form $j_\alpha(z) \quad \hat{\tilde{j}}_\alpha(\bar{z})$

but only $\epsilon^\alpha \left(j_\alpha(z) + \tilde{j}_\alpha(\bar{z}) \right)$

is a conserved $\Delta = 1$ primary. So:

Thm 3: In the open case $\mathcal{A}^{\pm}_{[0,\pi]}$,

$\epsilon^\alpha \int (j_\alpha + \tilde{j}_\alpha)\, d\sigma$ generates an action of $SP_{16}(1,9)$.

Now again we form BRST coho — (unitarity ⇒)

Lie algebra coho for superconformal algebra

$$\mathcal{H} = H^*_d(\mathcal{A}^+_{NS} \oplus \mathcal{A}^+_R)$$

on $\mathcal{A}_R$, d = Dirac – Ramond operator,

$= \not{\partial}$ on $\mathcal{L}\mathbb{R}^{1,9}$

$m^2 = 0$ Ind $(8_v \oplus 8_+) \otimes u(N)$, VM of $S\mathcal{P}_{16}(1,9)$

Fields: $A \in \mathcal{Q}(\mathbb{R}^{1,9}; u(N))$ $\chi \in \Gamma(16_- \otimes u(N))$

Remark: on $\mathcal{A}^+_R|_{N=0}$ $d = p_\mu \psi_0^\mu$ = <u>Dirac operator</u>

Low energy: $S = \int \mathrm{Tr}(F^2 + \bar{\chi} \not{D} \chi) + \cdots$

———— x ————

<u>Closed sector</u>

$$\mathcal{A}_{closed} = (\mathcal{A}^+_{NS} \oplus \mathcal{A}^\pm_R) \otimes (\tilde{\mathcal{A}}^+_{NS} \oplus \tilde{\mathcal{A}}^+_R)$$

+ : IIB — : IIA

- Action of $S\mathcal{P}_{16_\pm}(1,9) \oplus S\mathcal{P}_{16_+}(1,9)$, 32 supercharges

(NS, NS) ⇒ G, B, ϕ as before

New spacetime bosons (R, R) of type spinor⊗spinor

$m^2 = 0$:

IIA $\quad 16_- \otimes 16_+ \cong [\Omega^{even}(\mathcal{S})]^+ \ni G$

IIB $\quad 16_+ \otimes 16_+ \cong [\Omega^{odd}(\mathcal{S})]^+ \ni \underbrace{G}_{\text{fields}}$

BRST conditions

ker d : $\quad d = \not{\partial} \longrightarrow$ exterior $\therefore \quad dG = 0$, $\; d{*}G = 0$

im d : $\quad G$ gauge invariant

Conclusion: RR sector is a generalized Maxwell theory of forms

$$G^{(p+2)} = d\,C^{(p+1)}, \quad C^{(p+1)} \in \Omega^{p+1}$$

potential

$$C^{(p+1)} \to C^{(p+1)} + d\,\Lambda^{(p)}$$

Bianchi: $\quad d\,G^{(p+2)} = 0$

EOM $\quad d * G^{(p+2)} = 0$

IIA : $\quad C^{(1)}, \; C^{(3)}$

IIB : $\quad C^{(0)}; \; C^{(2)}; \; C^{(4)}$

$$\mathcal{L}_{sugra} = \int e^{-2\phi}\left(\sqrt{g}\,R + |d\phi|^2 + |dB|^2\right)$$

$$+ \int \sum_{p=0,2} dC^{(p+1)} \wedge * dC^{(p+1)}$$

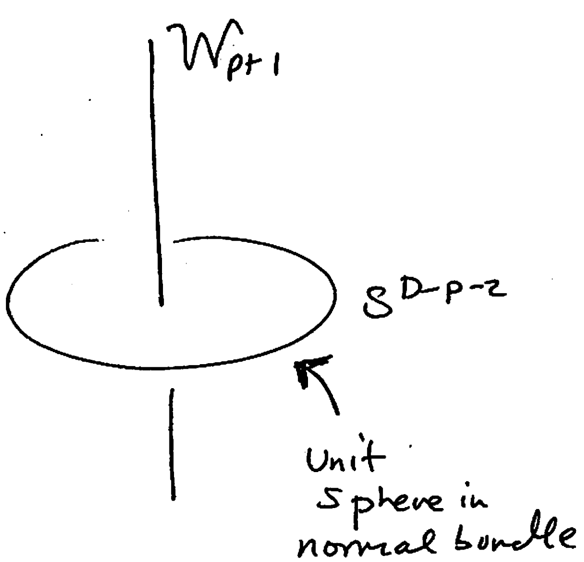
keep this

Admits soliton solutions

$$d * G^{(p+2)} = q_e\, \eta\left(W_{p+1} \hookrightarrow \mathbb{R}^{1,9}\right)$$

η : δ-function supported representative of P.P.

These solitons have topological R.R. charge



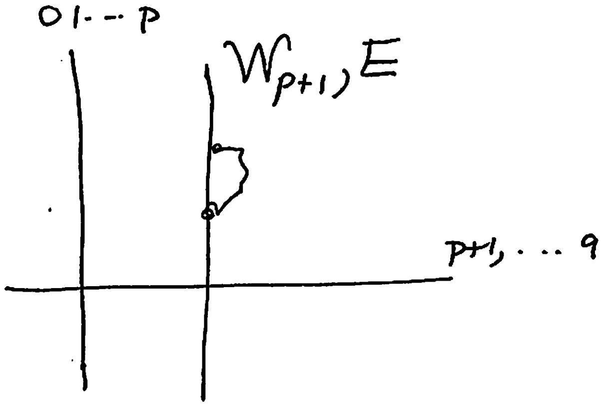
$$q_e = \int * G^{(p+2)} \neq 0$$

To acct. for source –

Add to action: $q_e \int_{W_{p+1}} C^{(p+1)}$ } this and its generalization will be <u>important</u>.

D branes as BPS States

Now include: (D,D) & (N,N) sectors



Lorentz symmetry: $Spin(1,9) \supset \overbrace{Spin(1,p)}^{\text{Lorentz}} \times \overbrace{Spin(9-p)}^{\text{global, "R-symmetry"}}$

Because oscillators unchanged: $SP_{16}(1,p)$

the extended superPoincaré algebra with 16 supercharges acts on $\mathcal{H}_{\mathcal{B},\mathcal{B}}$

e.g. $p=3$ $\quad 16_+ \to S^+ = (2;1;4) \oplus (1,2;\bar{4})$ $\quad Spin(1,3) \times Spin\, 6$

Fields $\quad A, \phi_a \quad \chi \in \Gamma(S^- \otimes u(N))$

$$\frac{1}{g_{YM}^2} \int_{W_{p+1}} tr F^2 + tr (D\phi)^2 + tr [\phi,\phi]^2 + tr\, \bar{\chi} \not{D} \chi + \cdots$$

$\longrightarrow$ YM (W_{p+1}, E) , as before

But now we come to the more subtle point of how to interpret the state Ψ in the 10D sugra theory.

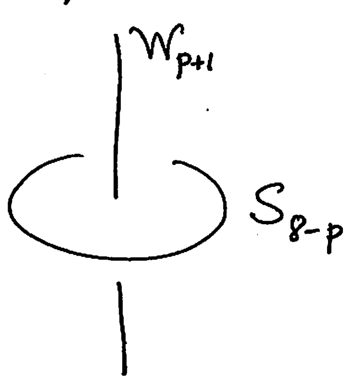
There are two key statements

A.) If the $U(N)$ SYM is in its groundstate $F(A)=0$, $V(\phi)=0$ then

$$\left(\epsilon^{\alpha} Q_{\alpha} + \tilde{\epsilon}^{\alpha}\tilde{Q}_{\alpha}\right)|\Psi(A,\phi,\cdots)\rangle = 0$$

for $* \boxed{\tilde{\epsilon} = \Gamma^{01\cdots p}\,\epsilon}$, hence it is BPS.

B.) If the SYM is in its gndstate then



$$\int_{S_{8-p}} * G^{(p+2)} = N$$

i.e. it is electrically charged under $C^{(p+1)}$ hence at long distances, appears as an electrically charged topol. soliton.

Justify A:

Theorem: $\mathcal{A}(\mathcal{B},\mathcal{B})$ has a conserved spacetime supercurrent $\epsilon^{\alpha} j_{\alpha}(z) + \tilde{\epsilon}^{\alpha} \tilde{j}_{\alpha}(\bar{z})$ for $(*)$

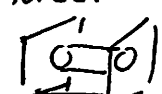
Pf: All (N,N) $\epsilon^{\alpha} = \tilde{\epsilon}^{\alpha}$

go to (D,D) by E–M duality on w.s.

$$(\tilde{X}^{\mu}, \tilde{\psi}^{\mu}) \longrightarrow -(\tilde{X}^{\mu}, \tilde{\psi}^{\mu})$$

Represented on spinors by Γ^{μ}. ▨

Corollary: no force:



confirms that from ↑

Of cource, you can compute long distance fields, ~~a~~ ~~idealtele~~... but ∃ slicker argument

Justify B:

It is not surprising that susy's are broken since

$$\{Q_{\alpha}, Q_{\beta}\} = (C\Gamma^{M})_{\alpha\beta} P_{M}$$

and P is broken –

What is much more surprising is that any are preserved at all!

In the nonptve theory the susy algebra is modified

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M + (C\Gamma^{M_1\cdots M_p})_{\alpha\beta} Z_{M_1\cdots M_p}$$

$Z \neq 0$ when $\exists$ p-brane electrically charged under $C^{(p+1)}$

For this algebra $32 \rightarrow 16$ with positive energy.

~~[illegible]~~
Thus Dp-branes in their SYM groundstate provide the microscopic description of the BPS (RR) solitons of type II sugra.

Emerging picture of Bulk + brane Lagrangian

$$\left[\frac{1}{l_s^8} \int e^{-2\phi} \left(\sqrt{g} R + |d\phi|^2 + |dB|^2 \right) + \cdots \right.$$

$$\left. + \int \sum_p \quad \cdots \right.$$

as above, but now we put $q_e = N = ch_0(E)$

$$+ \frac{1}{g_{YM}^2} \int_{W_{p+1}} \mathrm{Tr}\, F^2 + (D\phi)^2 + ([\phi,\phi])^2 + \bar{\chi} \not{D} \chi$$

$$+ \int_{W_{p+1}} z^* C^{(p+1)} \cdot ch_0(E)$$

Now we want to start asking – What happens when the SYM is not in its groundstate? In general the answer is very complicated, but for certain states we can say something.

10. Instantons as solitons in $YM(\mathcal{B}_{p+4})$

We are now finally ready to start considering Yang-Mills instantons.

We consider a $D(p+4)$-brane whose worldvolume includes the directions $\mathbb{R}^4_{6789}$. The low energy theory is a Yang-Mills theory $YM(\mathcal{B}_{p+4}, E)$. We now consider a Chan-Paton bundle

$$E \to \mathcal{W}_{p+5}$$

which is topologically nontrivial (with L^2 conditions at ∞).

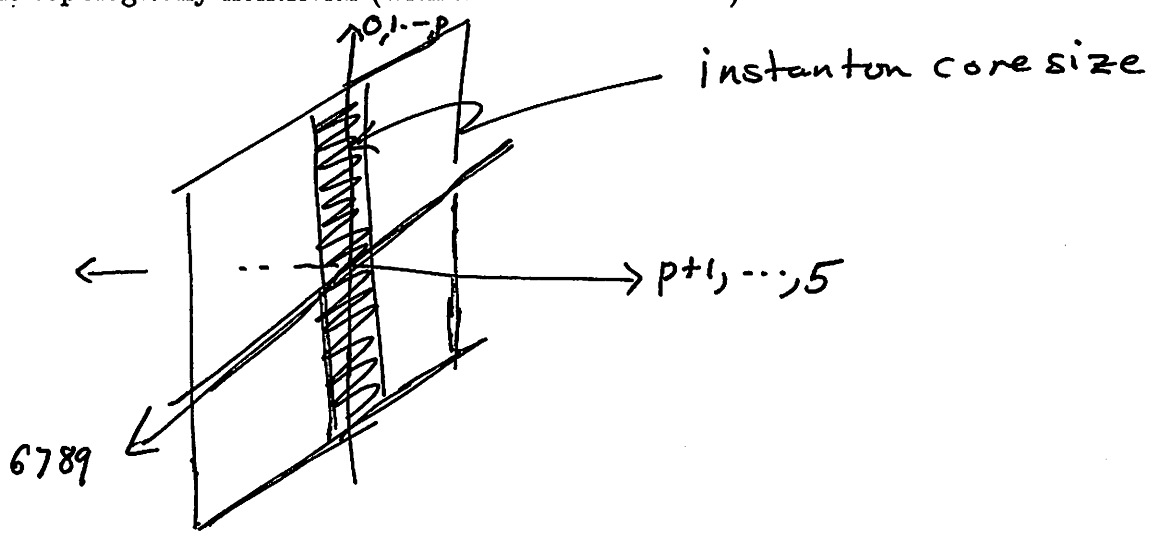


Fig. 24: A Yang-Mills instanton in the theory $YM(\mathcal{B}_{p+4})$.

Choose a coordinate subspace $\mathcal{B}_{p+4} = \mathcal{B}_p \times \mathbb{R}^4_{6789}$

The Yang-Mills instantons on $\mathbb{R}^4_{6789}$ which are translationally invariant (in $0,1,\cdots,p$) define p-dimensional solitons in $YM(\mathcal{B}_{p+4}, E)$. The energy density is proportional to the action density for the 4d yang-mills theory on $\mathbb{R}^4_{6789}$.

Note - the mathematical instanton defines a p-brane physical object in the SYM theory $YM(\mathcal{B}_{p+4})$

$p = 0$ particle

$p = 1$ string

$p = 2$ membrane

Let us call this state $|\Psi(A)\rangle_{YM(\mathcal{B}_{p+4})}$.

The solitonic state $|\Psi(A)\rangle_{YM(\mathcal{B}_{p+4})}$ is a BPS state in the SYM theory.

Proof:

L3-14

The instanton satisfies the equations:

$$F_{MN} = 0 \qquad M, N \notin \{6,7,8,9\}$$

$$F = *F \qquad M, N \in \{6,7,8,9\}$$

Consider the supersymmetry variation for a spinor ϵ of $Spin(1,p) \times Spin(d_T)$:

$$\begin{aligned} \delta\chi &= \Gamma^{MN}\epsilon F_{MN} \\ &= \frac{1}{2}\left(\Gamma^{MN}F_{MN} + \Gamma^{MN}\Gamma^{6789}\tilde{F}_{MN}\right)\epsilon \\ &= \Gamma^{MN}F_{MN}\left(1 + \Gamma^{6789}\right)\epsilon \end{aligned}$$

So if

$$\epsilon = -\Gamma^{6789}\epsilon \tag{11.1}$$

then for the state $|\Psi(A)\rangle_{YM(\mathcal{B}_{p+4})}$

$$\delta_\epsilon\chi = \langle\Psi(A)|\{\epsilon\cdot Q, \chi\}|\Psi(A)\rangle = 0 \tag{11.2}$$

That is: spinors satisfying (11.1) define supercharges $\epsilon \cdot Q$ in the QFT $YM(\mathcal{B}_{p+4})$ which annihilate the instantonic-soliton. Therefore, these solitons preserve at least (in general, exactly) half the supersymmetry.

Moreover, there is a moduli space of these BPS states $\mathcal{M}_{BPS} = \mathcal{M}^+ = \{A \mid F^+(A)=0\}/\mathcal{G}$.

11.1. Brane charges and instantonic solitons

Let us now return to the anomalous couplings on a Dp-brane.

As a special case of (10.3) we get the interaction:

$$\int_{\mathcal{W}_{p+5}} C^{(p+4)}ch_0(\mathcal{F}) + C^{(p)}ch_2(\mathcal{F})$$

Let us choose a hyperplane $\mathcal{B}_p \subset \mathcal{B}_{p+4}$ such that:

$$\mathcal{B}_{p+4} = \mathcal{B}_p \times \mathbb{R}^4_{6789}$$

Let us furthermore consider a gauge field configuration in the theory $YM(\mathcal{B}_{p+4})$ such that

$$\int_{\mathbb{R}^{6789}} \mathrm{Tr} F \wedge F = 8\pi^2 v$$

A $(p+4)$-brane with a $U(w)$ brane gauge field with instanton number v i.e., a Chan-Paton bundle with characteristic classes: $(ch_0 = w, ch_2 = v)$ has the same RR charge as a composite of w $(p+4)$-branes and v p-branes.

This suggests that just the way Dp-branes are the microscopic description of SUGRA solitons, "branes within branes" give the microscopic description of the instantonic solitons on Dp-branes.

In the next sections we will verify that this is correct.

later

123-15

How gauge fields change

10. The charge of a Dp-brane

We argued that D-branes carry charge. This means that the SUGRA action in the presence of a brane must have something like the form:

$$\frac{1}{2}\int_{\mathbb{R}^{1,9}} dC^{(p+1)} \wedge *dC^{(p+1)} + \mu_p \int_{\mathcal{W}_{p+1}} C^{(p+1)}$$

and the presence of the brane modifies the equations to:

$$\begin{aligned} dG^{(p+2)} &= 0 \\ d * G^{(p+2)} &= (-1)^{p+1}\mu_p \eta(\mathcal{W}_{p+1} \hookrightarrow \mathcal{S}) \end{aligned} \qquad (10.1)$$

μ_p is the charge of the Dp-brane. Evidently, it should be proportional to the rank N of the CP bundle $E \to \mathcal{W}_{p+1}$.

10.1. Modification in the presence of gauge fields on the brane

The presence of gauge fields on the brane modifies (10.1) in an important way.

We want to dodge several tricky issues so we just state what we believe is the correct result.

Let:

$G = \sum_p dC^{(p+1)}$ be the total RR fieldstrength

$\iota : \mathcal{W}_{p+1} \hookrightarrow \mathcal{S}$ the embedding of the brane worldvolume into spacetime.

$E \to \mathcal{W}_{p+1}$ the CP bundle, with typical fiber $V \cong \mathbb{C}^N$.

$\mathcal{F} \equiv F + \iota^*(B)1_N$ the modified fieldstrength.

$\mathrm{ch}\mathcal{F} \equiv Tr_V \exp(\frac{i\mathcal{F}}{2\pi})$

Then we claim that the equation of motion and Bianchi identity are unified into the single equation:

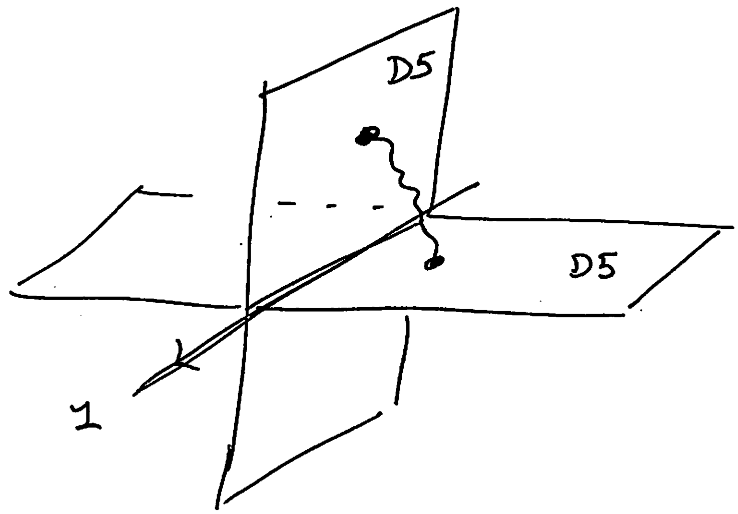
$$dG = \iota_*(\mathrm{ch}\mathcal{F}) \qquad (10.2)$$

where ι_* is the push-forward defined using Poincare duality.

Heuristically, we write this in terms of δ-functions over the coordinates x^i transverse to $\mathcal{W}$:

$$dG = \prod_{i=1}^{d_T} \delta(x^i)dx^i \cdot Tr_V \exp(\frac{i\mathcal{F}}{2\pi})$$

The derivation of (10.2) is based on the "inflow argument" associated with a configuration of orthogonally intersecting D-branes:



chiral fermions at the intersection have an anomaly which is cancelled by the C.S. coupling:

$$\int_{W_{p+1}} \imath^* C \wedge ch\, \mathcal{F}$$

∃ many subtleties here.

Special case:

$$\int_{W_{p+5}} C^{(p+4)} ch_0(\mathcal{F}) + C^{(p)} ch_2(\mathcal{F})$$

If we have a $U(w)$ instanton $A_\mu(x)$ in $YM(\mathcal{B}_{p+4}, E)$ invariant along $\mathcal{B}_p$, varying in $\mathbb{R}^4_{6789}$

$$\begin{cases} \int_{\mathbb{R}^4_{6789}} \mathrm{Tr}\, F\wedge F = 8\pi^2 v \\ ch_0 E = w \end{cases}$$

then this state

$$\left| \Psi(W_{p+5}, E, A_\mu(x), \dots) \right\rangle_{S.F.T.\ (\mathbb{R}^{1,9})}$$

has RR charge of a <u>composite</u> system:

$$(\mathcal{B}_{p+4}, \mathbb{C}^w) \parallel (\mathcal{B}_p, \mathbb{C}^v)$$

Now – Dp branes give BPS solitons with (R,R) Charge – Occam's razor suggests the converse. We will want to identify...

[L3-18

12. The $(p, p+4)$ system

Let us now consider two parallel Dp branes.

$$(\mathcal{B}'_{p+4}, \mathbb{C}^w) \parallel (\mathcal{B}_p, \mathbb{C}^v)$$

Note there are two cases:

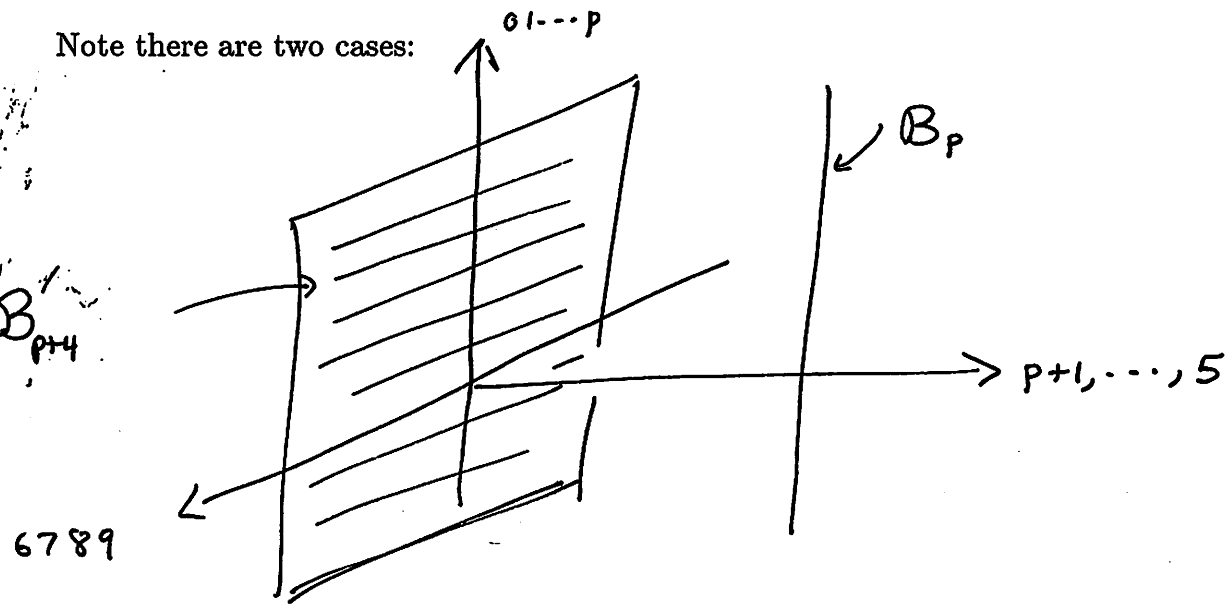


Fig. 25:

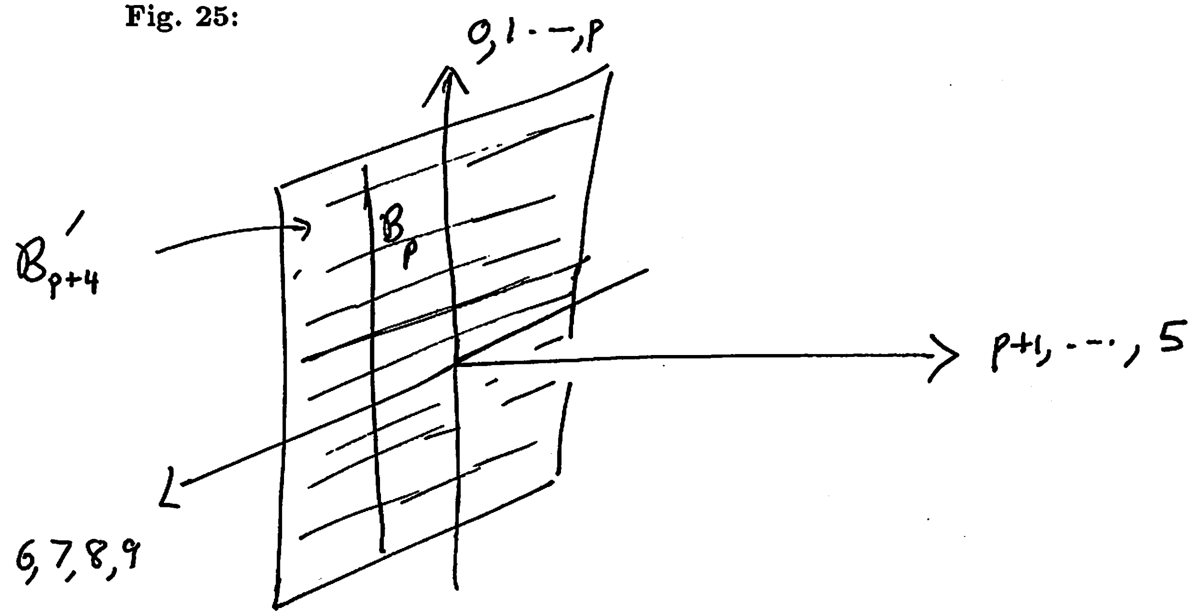


Fig. 26: $\mathcal{B}_p$ can lie within or outside $\mathcal{B}' = \mathcal{B}_{p+4}$.

For the moment take $\mathcal{B}_p \subset \mathcal{B}'_{p+4}$. Indeed, let us take $p = 5$, so this is the only possibility.

Let us find the global symmetries

$$\mathcal{G} = Spin(1,5) \times (Spin\,4)_{6789} \times U(v) \times U(w)$$

$$\mathcal{G}_{little} = (Spin\,4)_{2345} \times \;''\; \times \;''\; \times \;''$$

Unbroken susy's

$$\tilde{\epsilon} = \Gamma^{01\cdots p}\epsilon \qquad \mathcal{B}_p$$

$$= \Gamma^{01\cdots p}\Gamma^{6789}\epsilon \qquad \mathcal{B}_{p+4}$$

first eliminates $\tilde{\epsilon}$. Second defines $\dim_{\mathbb{R}} = 8$

subspace $S^+ \subset 16_+$ by $\epsilon = \Gamma^{6789}\epsilon$

$\longrightarrow$ Family of 8-supercharge algebras $\mathcal{SP}_8(1,p)$

e.g. $p = 5$ $\qquad S^+ = (4; 2,1)_{\mathbb{R}} + (\bar{4}; 2,1)_{\mathbb{R}}$ $\quad Spin(1,5) \times Sp$

minimal susy in $\mathbb{R}^{1,5}$.

Now we search for the spacetime field theory on the brane $\mathcal{B}_p$.

First recall:

2 rep's of: $SO_8(1,5) \times \underbrace{U(V)}_{\text{gauge}}$

characterized by bosonic rep$^{\underline{n}}$ of little group

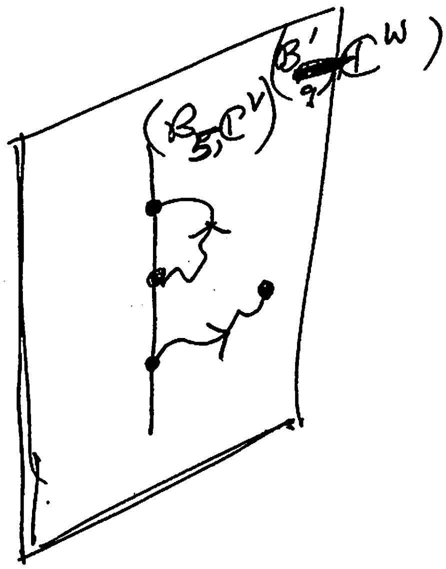
$$(Spin\,4)_{2345} \times (Spin\,4)_{6789} \times U(V):$$

VM : $(2,2;1,1) \otimes u(V)$

HM : $\left[(1,1;2,1) \otimes (R \oplus R^*)\right]_{\mathbb{R}}$

R = complex $\overbrace{\text{rep}^{\underline{n}}}^{\text{hermitian}}$ of $U(V)$

Note $R \oplus R^*$ is a quaternionic v.s.



$$\mathcal{A}(\mathcal{B}_5, \mathcal{B}_5) \longrightarrow \begin{cases} 1\ \text{VM} & A_{\mu = 0,\cdots 5} \text{ of } U(V) \\ 1\ \text{HM} & \phi_{6,7,8,9}, \quad \mathcal{R} = u(V) \end{cases}$$

Remark: together these make 1 VM of $SP_{16}(1,5)$, but because of interactions —

$$\mathcal{A}(\mathcal{B}_5, \mathcal{B}_9) \longrightarrow \quad 1\ \text{H.M. with } \mathcal{R} = (\bar{V}, W) \text{ of } U(V) \times U(W)$$

Fields: $\tilde{h}^{Am}_{(\alpha)\ i}$ $\quad A = 1,2$

$h^{Ai}{}_{m}$ $\quad m = 1, \ldots, w$

$\quad i = 1, \ldots, v$

Remark: Spinor index A on bosonic fields from (D,N)

The $\tilde{h}$ are complex fields and the spinor index shows that we actually are getting quaternions.

* Now we quote a theorem of supersymmetry

Most general 2-derivative lagrangian of $SP_8(1,5)$ with VM's + linear HM's is completely specified by the data

- $G \longrightarrow$ VM, bilinear form on $\mathfrak{g}$
- quaternionic rep$^{\underline{n}}$ $R \oplus R^*$ of G
- $\vec{\zeta} \in \mathrm{Center}(\mathfrak{g}) \otimes \mathbb{R}^3$

$$L_{Bos} = \frac{1}{g^2_{YM}} \int d^6\xi \left(\mathrm{Tr}(F^2_{\mu\nu} + \cdots) + (Dh, Dh)_R + (\vec{\mu} - \vec{\zeta})^2 \right)$$

Moduli space of supersymmetric vacua is the H.K. quotient:

$$\mathcal{M}_{vacua} = \{ \phi \mid \vec{\mu}(\phi) = \vec{\zeta} \} / G.$$

Now we apply this to the brane theory $YM(\mathcal{B}_5, \mathbb{C}^V)$.

Can show $\vec{\zeta} = 0$

$$\mathcal{M}_{vac.} = \{ (\phi, h, \tilde{h}) \mid [\phi_\ell, \phi_{\ell'}]^i{}_j \, \sigma_{AB}{}^{\ell\ell'} + (\tilde{h}_A)^i{}_m (h_B)^m{}_j + (\tilde{h}_B h) = 0 \} \mod U(V)$$

The identification

$$\underset{YM(\mathcal{B}_{p+4})}{\mathcal{M}_{BPS}} \cong \underset{YM(\mathcal{B}_p)}{\mathcal{M}_{vac}}$$

is the physical interpretation of the ADHM construction.

* Concluding remarks

(A) Extensions:

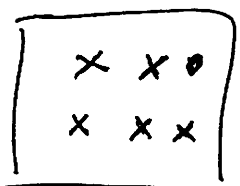
— pushing it further, you can recover the ADHM construction of the gauge field itself

— many aspects of instanton physics become transparent, e.g. Nahm transform,

$\mathbb{R}^5 \times T^4$ W $\mathcal{B}_9$

V $\mathcal{B}_5 \perp T^4$

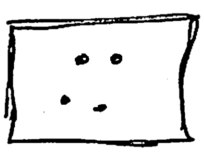
T^4



Recall T-duality rules from last time

↓ T duality

$\hat{T}^4$

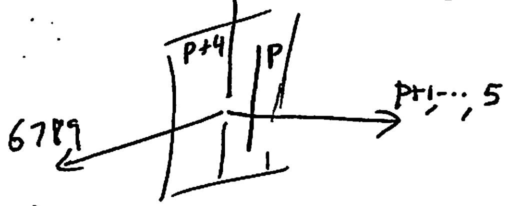


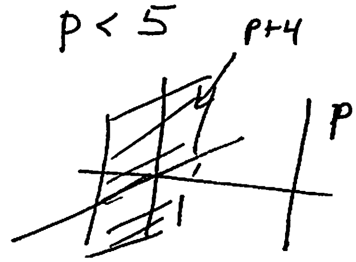
V $\mathcal{B}_9$

W $\mathcal{B}_5 \perp \hat{T}^4$

(B) Work for the future

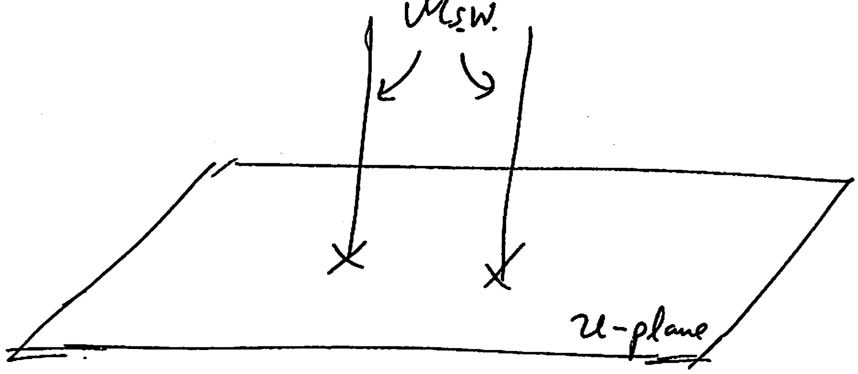
1. Coulomb + Higgs $p < 5$





can only happen with zero size

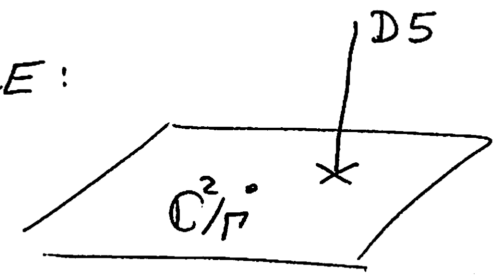
quite analogous to SW theory:



– in the string case the Coulomb phase is not yet completely understood – but it involves a very novel answer to "What happens to a small instanton"

2. Probing metrics –

Works well for ALE:



K3? black holes?

Prediction: D-brane techniques will give the first construction of a nontrivial metric on K3.